# Hamilton Weights and Petersen Minors

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**Abstract:** A (1, 2)-eulerian weight $w$ of a cubic graph is called a Hamilton weight if every faithful circuit cover of the graph with respect to $w$ is a set of two Hamilton circuits. Let $G$ be a 3-connected cubic graph containing no Petersen minor. It is proved in this paper that $G$ admits a Hamilton weight if and only if $G$ can be obtained from $K_4$ by a series of $\triangle \leftrightarrow Y$-operations. As a byproduct of the proof of the main theorem, we also prove that if $G$ is a permutation graph and $w$ is a (1,2)-eulerian weight of $G$ such that $(G, w)$ is a critical contra pair, then the Petersen minor appears ''almost everywhere'' in the graph $G$. © 2001 John Wiley & Sons, Inc. J Graph Theory 38: 197–219, 2001

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## 1. INTRODUCTION

We assume familiarity with graph theory. A *circuit* is a 2-regular connected graph. By $G \cong H$ we mean that the graphs $G$ and $H$ are isomorphic. A graph $H$ is called a *minor* of a graph $G$ if $H$ is isomorphic to a contraction of a subgraph of $G$. A Petersen minor is a minor isomorphic to the Petersen graph.

A path $P = v_1 \ldots v_p$ $(p \geq 3)$ of a graph $G$ is called *a subdivided edge* of $G$ if $d_G(v_i) = 2$ for each $v_i$ with $2 \leq i \leq p-1$. The *underlying graph* of a graph $G$, denoted by $\overline{G}$, is the graph obtained from $G$ by replacing each maximal subdivided edge with a single edge. Let $H_1$ and $H_2$ be two subgraphs of a graph $G$. The *symmetric difference* of $H_1$ and $H_2$, denoted by $H_1 \triangle H_2$, is the subgraph of $G$ induced by edges of $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$.

A map $w : F(G) \mapsto \{1, 2\}$ is called a (1, 2)-*weight* of $G$. A graph $G$ associated with a weight $w$ is usually denoted by an ordered pair $(G, w)$. A (1, 2)-weight of a graph $G$ is *eulerian* if the total weight of each edge-cut of $G$ is even. A weight $w$ of a graph $G$ is *admissible* if, for each edge-cut $T$ of $G$ and for each edge $e \in T$, the weight of $e$ is no more than one half of the total weight of $T$. (It is obvious that, for a (1,2)-eulerian weight $w$ of $G$, $G$ is bridgeless if and only if $w$ is admissible.) Let $G$ be a bridgeless graph and with a (1,2)-weight $w$. A family $\mathcal{F}$ of circuits of $G$ is called *a faithful circuit cover* of $(G, w)$ if each edge $e$ of $G$ is contained in precisely $w(e)$ members of $\mathcal{F}$.

Let $w$ be a (1, 2)-eulerian weight of a cubic graph $G$. A faithful circuit cover $\mathcal{F}$ of $(G, w)$ is called a *Hamilton cover* if $\mathcal{F}$ is a set of two Hamilton circuits. A $(1, 2)$-eulerian weight $w$ of $G$ is called a *Hamilton weight* if *every* faithful circuit cover of $(G, w)$ is a Hamilton cover. If $(G, w)$ has a faithful circuit cover which is not a Hamilton cover, then $(G, w)$ is *non-Hamilton coverable*.

A cubic graph $G$ is *uniquely edge-3-colorable* if there is only one way to partition $E(G)$ into three perfect matchings of $G$. In other words, $G$ has only one 1-factorization.

The $\triangle \rightarrow Y$-*operation* is an operation of a cubic graph that contracts a triangle to a vertex. *The* $Y \rightarrow \triangle$-*operation* is an operation of a cubic graph that expands a vertex to a triangle (see Fig. 1). It is well known that $\triangle \leftrightarrow Y$-operations on a cubic graph preserves the property of being uniquely edge-3-colorable.

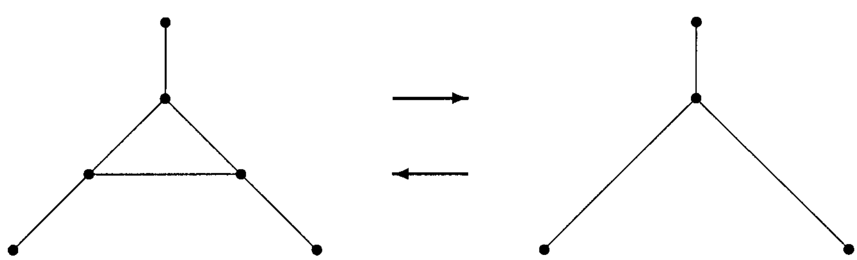

FIGURE 1. $Y \rightarrow \triangle$-operation and $\triangle \rightarrow Y$-operation.

Let $\langle\mathcal{K}_4\rangle$ be the set of all cubic graphs which can be obtained from $K_2^3$ by a series of $Y \to \triangle$ or $\triangle \to Y$ operations (where $K_2^3$ is the graph with 2 vertices and 3 parallel edges).

The study of Hamilton weights and Hamilton covers is motivated by and intimately related to the following well-known conjectures.

**Conjecture A** (Circuit double cover conjecture, Szekeres [16] and Seymour [15], or see [13]). *Every bridgeless graph has a family of circuits that covers each edge precisely twice.*

**Conjecture B** (Unique edge-3-coloring conjecture, Fiorini and Wilson [5,6]). *Let $G$ be a cubic planar graph. Then $G$ is uniquely edge-3-colorable if and only if $G \in \langle\mathcal{K}_4\rangle$.*

Prior studies on the problem of unique edge-3-colorings of cubic graphs can be seen, for example, in [5,6,10,11,17,18–20], among others.

It is well-known that a smallest counterexample $G$ to Conjecture A can be assumed to be cubic and cyclically 4-edge-connected (see [15] and [13]). Let $e \in E(G)$ and choose a double cover $\mathcal{F}$ of $G\setminus\{e\}$ such that $|\mathcal{F}|$ is as large as possible. It is evident that the underlying graph of $C \cup C'$, for each pair of adjacent members of $\mathcal{F}$, admits a Hamilton weight. Thus, it has been expected that the structure of $C \cup C'$ would provide some powerful tools to the final solution of Conjecture A and some other related circuit covering problems.

It was proved in [21] that a 3-connected cubic graph containing no Petersen minor and admitting a Hamilton weight must be uniquely edge-3-colorable. An analog of Conjecture B was also proposed in that paper (Conjecture 4.5 of [21]) that every 3-connected cubic graph $G$ admitting a Hamilton weight is in $\langle\mathcal{K}_4\rangle$. This conjecture is partially proved in this paper for Petersen minor free graphs (Theorem 1.1).

**Theorem 1.1.** *Let $G$ be a 3-connected cubic graph containing no Petersen minor. Then $G$ admits a Hamilton weight if and only if $G \in \langle\mathcal{K}_4\rangle$.*

It is necessary for $G$ to be 2-connected to admit a Hamilton weight. Let $(G_1, w_1)$ and $(G_2, w_2)$ be two cubic graphs with (1,2)-weights $w_1$ and $w_2$, respectively. For each $i \in \{1,2\}$, let $e_i = x_iy_i \in E(G_i)$ with $w_i(e_i) = 2$. Define a new graph $G_1 \circ_{(e_1,e_2)} G_2$ from the disjoint union of $G_1 - e_1$ and $G_2 - e_2$ by adding new edges $e_x = x_1x_2$ and $e_y = y_1y_2$. Then the following characterizes graphs admitting a Hamilton weight among 2-connected cubic graphs without a Petersen minor.

**Corollary 1.2.** *Let $G$ be a 2-connected cubic graph containing no Petersen minors. Then $G$ admits a Hamilton weight if and only if either $G \in \langle\mathcal{K}_4\rangle$ or for some integer $t \geq 1$, there exist graphs $G_1, G_2, \ldots, G_{t+1} \in \langle\mathcal{K}_4\rangle$ such that*

$$G = \left(\cdots\left(\left(G_1 \circ_{\left(e_1^{(1)},e_2^{(1)}\right)} G_2\right) \circ_{\left(e_1^{(2)},e_2^{(2)}\right)} G_3\right) \circ_{\left(e_1^{(3)},e_2^{(3)}\right)} \cdots\right) \circ_{\left(e_1^{(t)},e_2^{(t)}\right)} G_{t+1}.$$

The proof of the main results will be given in the last section. In Section 2, we present some preliminaries on $\triangle \leftrightarrow Y$-operations and on faithful circuit coverings. An associated result will be proved in Section 3, which is needed in Section 4.

## 2. PRELIMINARIES

Let $i \geq 0$ be an integer. For a $(1,2)$-weight $w$ of a graph $G$, denote

$$E_{w=1}(G) = \{e \in E(G) : w(e) = i\}.$$

When there is no confusion arises, we shall write $E_{w=i}$ for $E_{w=i}(G)$. Some straightforward properties about $\langle \mathcal{K}_4 \rangle$ are listed below, the proofs of which are omitted or sketched.

**Lemma 2.1.** *The $\triangle \leftrightarrow Y$-operation of a cubic graph preserves the property of admitting a Hamilton weight, and also preserves the number of 1-factorizations of the graph.*

**Lemma 2.2.** *Each $G \in \langle \mathcal{K}_4 \rangle$ has a unique 1-factorization.*

**Lemma 2.3.** *For a graph $G \in \langle \mathcal{K}_4 \rangle$ let $w$ be a (1, 2)-eulerian weight of $G$ with $E_{w=2} = M$. Then $w$ is a Hamilton weight of $G$ if and only if $M$ is a 1-factor with $|M \cap T| = 1$ for each 3-edge-cut $T$ of $G$.*

***Proof.*** Apply $\triangle \leftrightarrow Y$-operations and argue by induction on $|V(G)|$. ■

**Lemma 2.4** ([9]). *Every graph $G \in \langle \mathcal{K}_4 \rangle$ can be obtained from $K_2^3$ by a series of only $Y \rightarrow \triangle$ operations.*

**Definition 2.5.** *Let $n \geq 0$ and $k \geq 0$ be integers, and let $\phi : \{1, 2, \ldots, n\} \mapsto \{x, y, z\}$ be a function.*

*Define $\Gamma_1 = \Gamma_1(n, \phi)$ to be the graph obtained from the vertex disjoint paths $P_x = x_0x_1 \cdots x_nx'_0$, $P_y = y_0y_1 \cdots y_ny'_0$ and $P_z = z_0z_1z_2 \cdots z_nz'_0$ such that the vertex set and the edge set of $\Gamma_1$ are $V$ and $E$, respectively given below:*

$$\begin{aligned}
V &= V(P_x) \cup V(P_y) \cup V(P_z) \text{ and} \\
E &= E(P_x) \cup E(P_y) \cup E(P_z) \\
&\quad \cup \{x_0y_0, y_0z_0, z_0x_0, x'_0y'_0, y'_0z'_0, z'_0x'_0\} \\
&\quad \cup \{x_iy_i | 1 \leq i \leq n \text{ and } \phi(i) = z\} \\
&\quad \cup \{y_iz_i | 1 \leq i \leq n \text{ and } \phi(i) = x\} \\
&\quad \cup \{z_ix_i | 1 \leq i \leq n \text{ and } \phi(i) = y\}.
\end{aligned}$$

*Let $\Gamma(n, \phi)$ be the underlying cubic graph of $\Gamma_1$. Note that the range of $\phi$ has only one element if and only if there is an edge joining the two triangles $x_0y_0z_0x_0$ and $x'_0y'_0z'_0x'_0$ in $\Gamma(n, \phi)$.*

*For $m \geq 3$, let $\Lambda(m)$ be the graph obtained from a circuit $C_{2m} = v_1 v_2 \cdots v_{2m} v_1$ by adding the new edges $\{v_1 v_{m+1}\} \cup \{v_i v_{2m+2-i} | \; 2 \leq i \leq m-1\}$.*

**Lemma 2.6.** *Let $G \in \langle \mathcal{K}_4 \rangle \backslash \{K_2^3, K_4\}$. Each of the following holds*:

(1) *G has at least two triangles.*
(2) *Every pair of distinct triangles of G are disjoint.*
(3) *If, in addition, G has exactly two triangles, then $G \cong \Gamma(n, \phi)$, for some n and $\phi$.*
(4) *If $G \cong \Gamma(n, \phi)$ for some $n \geq 0$ and $\phi$, and if the range of $\phi$ has only one element, that is, the only two triangles are joined by an edge, then $G \cong \Lambda(n+3)$.*

***Proof.*** Parts (1) and (2) follow from Lemma 2.4, and Part (4) is immediate from the definition of $\Gamma(n, \phi)$. It suffices to apply induction to prove Part (3).

Suppose that $T_1$ and $T_2$ are the only two triangles of $G$ and let $G_1$ be the graph obtained from $G$ by contracting $T_1$. If $G_1 \cong K_4$, then $G \cong \Gamma(0, \phi)$, the only 6-vertex graph in $\langle \mathcal{K}_4 \rangle$. Hence we may assume that $G_1 \neq K_4$. By induction, $G_1 \cong \Gamma(n-1, \phi_1)$ for some $\phi_1$. We shall use the notation for this $\Gamma$ in the definition above.

Let $v_T$ denote the contraction image of the triangle $T_1$ in $V(\Gamma)$. Then we may assume that $v_T \in \{x'_0, y'_0, z'_0\}$. Without loss of generality, we assume that $v_T = y'_0$. Thus $G \cong \Gamma(n, \phi)$ where $\phi(i) = \phi_1(i)$ if $i < n$, and $\phi(n) = y$, and so Part (3) is proved by induction. ■

**Lemma 2.7.** *Let $G_1$ and $G_2$ be connected cubic graphs, and for $i \in \{1, 2\}$, let $v_i \in V(G_i)$ be a vertex with neighbors $x_i, y_i, z_i$ in $G_i$. Obtain a new graph $G = G_1 \circ_{(v_1, v_2)} G_2$ from the disjoint union of $G_1 \backslash v_1$ and $G_2 \backslash v_2$ by adding the new edges $x_1x_2, y_1y_2, z_1z_2$. Then $G \in \langle \mathcal{K}_4 \rangle$ if and only if both $G_1$ and $G_2$ are in $\langle \mathcal{K}_4 \rangle$.*

***Proof.*** We argue by induction on $|V(G)|$. The lemma holds trivially if both $G_1, G_2 \in \{K_2^3, K_4\}$ for sufficiency and if $|V(G)| \leq 6$ for necessity.

Assume that $G_1 \in \langle \mathcal{K}_4 \rangle \backslash \{K_2^3, K_4\}$. Then $G_1 \backslash v_1$ has a triangle $T$. Apply a $\triangle \to Y$-operation to $G_1$ to get $G'_1$ by contracting $T$. By induction, $G' = G'_1 \circ_{(v_1, v_2)} G_2 \in \langle \mathcal{K}_4 \rangle$. But then $G$ can be obtained from $G'$ by applying a $Y \to \triangle$-operation to restore $T$, and so $G \in \langle \mathcal{K}_4 \rangle$. This proves the sufficiency.

If $G \in \langle \mathcal{K}_4 \rangle$, then $G$ has a triangle $T$. Since $E(T) \cap \{x_1x_2, y_1y_2, z_1z_2\} = \emptyset$, we may assume that T is a subgraph of $G_1$. Apply $\triangle \to Y$-operation to $G$ by contracting $T$, obtaining a graph $G'$. Then the same $\triangle \to Y$-operation may be also applied to $G_1$, resulting a graph $G'_1$ with $G' = G'_1 \circ G_2$. Since $G \in \langle \mathcal{K}_4 \rangle$, $G' \in \langle \mathcal{K}_4 \rangle$ also. By induction, both $G'_1$ and $G_2$ are in $\langle \mathcal{K}_4 \rangle$. Apply a $Y \to \triangle$-operation to $G'_1$ to restore $T$, we see that $G_1 \in \langle \mathcal{K}_4 \rangle$, and so this proves the necessity. ■

The same argument used by Seymour in the proof for (3.5) of [15] can be adopted to prove the following lemma.

**Lemma 2.8** ([15]). *Let w be a* $(1,2)$*-eulerian weight of a* 2*-connected cubic graph G. If* $E_{w=1}(G)$ *is a Hamilton circuit of G, then (G, w) has a faithful circuit cover.*

**Theorem 2.9** ([1] or [2]). *Let w be a* $(1,2)$*-eulerian weight of a* 2*-connected cubic graph G. If G contains no Petersen minor, then (G, w)* has a faithful circuit cover.

**Lemma 2.10.** *Let G be a cubic graph admitting a Hamilton weight w. Let* $\{H_1, H_2\}$ *be a faithful Hamilton cover of (G, w).* Then each of the following holds:

(1) *For each circuit C of* $E_{w=1}$, $\{H_1 \triangle C, H_2 \triangle C\}$ *is also a faithful Hamilton cover of* $(G, w)$.
(2) *If, in addition, G has a* 2*-edge-cut T, then* $T \subseteq E_{w=2}$ *and* $E_{w=1}$ *has more than one component.*

***Proof.*** (1) For each edge $e \notin E(C)$, $e$ is contained in $H_i$ if and only if $e$ is contained in $H_i \triangle C$, for each $i \in \{1, 2\}$. For each $e \in E(C)$, $e$ is contained in $H_i$ if and only if $e$ is contained $H_j \triangle C$, for each $\{i, j\} = \{1, 2\}$ since $e$ is contained in only one of $\{H_1, H_2\}$, not both. Thus, $\{H_1 \triangle C, H_2 \triangle C\}$ is a faithful cycle cover of $(G, w)$. Since $w$ is a Hamilton weight, $\{H_1 \triangle C, H_2 \triangle C\}$ is a Hamilton cover.

(2) Since $w$ is an eulerian weight, both edges in the cut $T$ are of the same weight. Assume that $T \subseteq E_{w=1}$. Then $T$ is contained in one of $\{H_1, H_2\}$, but not both. Say, $T \subseteq E(H_1)$ and $T \cap E(H_2) = \emptyset$. But $H_2$ cannot be a Hamilton circuit since it contains vertices of both components of $G \backslash T$ and does not pass through the cut $T$. ■

**Definition 2.11.** *Let x and y be two vertices of a graph G. A family* $\mathcal{C}$ *of circuits of G is called a circuit chain joining x and y if* $\mathcal{C} = \{C_1, \ldots, C_p\}$ *such that*

(i) $x \in V(C_1)$ *and* $y \in V(C_p)$.
(ii) $V(C_i) \cap V(C_j) \neq \emptyset$ *if and only if* $i = j \pm 1$.

**Lemma 2.12** ([21], or see [22]). *Let w be a Hamilton weight of a cubic graph G. Suppose that the graph G is* 3*-connected and contains no subdivision of the Petersen graph. Then, for each* $e_0 = xy \in E(G)$ *with* $w(e_0) = 2$*, every faithful circuit cover of* $(G \backslash \{e_0\}, w)$ *is a circuit chain joining x and y.*

***Proof.*** For each $e_0 = xy \in E_{w=2}$, by Theorem 2.9, $(G \backslash \{e_0\}, w)$ has a faithful circuit cover $\mathcal{F}$. Let $\mathcal{P} = \{C_1, \ldots, C_r\}$ be a circuit chain (each $C_i \in \mathcal{F}$) joining the vertices $x$ and $y$ in $G$. Let $H$ be the graph induced by edges covered by circuits of $\mathcal{P}$ and the edge $e_0$, and let $w'$ be a $(1,2)$-eulerian weight on $E(H)$ such that $w'(e)$ is the number of circuits of $\mathcal{P}$ containing the edge $e$ for each $e \neq e_0$, and $w'(e_0) = 2$. By Theorem 2.9, $H$ has a faithful circuit cover $\mathcal{F}'$. Then $\mathcal{F}' \cup [\mathcal{F} \backslash \mathcal{P}]$ is a faithful circuit cover of $G$. If $\mathcal{F} \neq \mathcal{P}$, then the faithful circuit cover $\mathcal{F}' \cup [\mathcal{F} \backslash \mathcal{P}]$ of $(G, w)$ is not a Hamilton cover, contrary to the assumption that $w$ is a Hamilton weight. Therefore $\mathcal{F} = \mathcal{P}$. ■

**Lemma 2.13.** *Let $(G, w)$ be a cubic graph with a (1,2)-eulerian weight, let $e = xy \in E(G)$ be an edge with $w(e) = 2$ and let $G' = \overline{G \setminus \{e\}}$. Suppose that $(G \setminus \{e\}, w)$ has a faithful circuit cover $\mathcal{F} = \{C_1, C_2, \cdots, C_t\}$ which is a circuit chain joining x and y. Then each of the following holds.*

(i) *Each component of $E_{w=1}(G')$ is an even circuit.*
(ii) *If, in addition, $E_{w=1}(G)$ is a Hamilton circuit C of G such that $C \cup \{e\}$ is a union of 2 even (odd, respectively) circuits whose intersection is $\{e\}$, then t is odd (even, respectively).*

***Proof.*** Let $X_0 = \{f \in E(C_j) : j \text{ is even}\}$ and let $X_1 = \{f \in E(C_j) : j \text{ is odd}\}$. Then color the edges in $X_0 \setminus X_1$ red, the edges in $X_1 \setminus X_0$ blue, and the edges in $X_0 \cap X_1$ yellow. This defines a proper 3-edge-coloring of the underlying graph $G' = \overline{G \setminus \{e_0\}}$. Since each component of $E_{w=1}(G')$ is alternatively colored red and blue, it must be of even length. This proves (i).

Now assume that $E_{w=1}(G)$ is a Hamilton circuit $C$. Let the subdivided edges of $G \setminus \{e\}$ that contain the endvertices of $e$ be $e'$ and $e''$. Suppose that $C \cup \{e\}$ is a union of two circuits of even (odd, respectively) lengths whose intersection is $\{e\}$. Since the edges in $E(C) = E_{w=1}(G)$ are alternatively red and blue colored, $e'$ and $e''$ must receive same (different, respectively) colors. Note that one of $\{e', e''\}$ is in $C_1 \setminus \cup_{i \geq 2} E(C_i)$ and the other is in $C_t \setminus \cup_{i \leq t} E(C_i)$, and so $t$ must be odd (even, respectively). ■

**Lemma 2.14** ([21], or see [22]). *If a 3-connected cubic graph G admits a Hamilton weight w and contains no subdivision of the Petersen graph, then the subgraph of G induced by edges of $E_{w=1}$ is a Hamilton circuit.*

***Proof.*** Let $w$ be a Hamilton weight of $G$ and $\{H_1, H_2\}$ be a Hamilton cover of $(G, w)$. Note that $\{H_1, H_2\}$ induces an edge-3-coloring $\{H_1 \setminus H_2, H_2 \setminus H_1, H_1 \cap H_2\}$ of $G$, and so each component of the 2-factor $E_{w=1} = H_1 \triangle H_2$ is a circuit of even length.

We want to show that $E_{w=1}$ has only one component. Since any two components of $E_{w=1}$ are joined in $G$ by an edge in $E_{w=2}$, it suffices to show that if $e_0 = xy \in E_{w=2}$, then $x$ and $y$ must be in the same component of $E_{w=1}$.

Since $G$ contains no subdivisions of the Petersen graph, by Theorem 2.9, $(G \setminus \{e_0\}, w)$ has a faithful circuit cover $\mathcal{F} = \{C_1, \ldots, C_r\}$, which, by Lemma 2.12, is also a circuit chain joining $x$ and $y$. By Lemma 2.13 (i), each component of $E_{w=1}$ not containing $x$ nor $y$ is an even circuit. If $x$ and $y$ are contained in different circuits of $E_{w=1}$, then $E_{w=1}$ would have at least two circuits of odd lengths, contrary to the fact that every circuit in $E_{w=1}$ has even length. Hence $x$ and $y$ are in the same component of $E_{w=1}$. Therefore $E_{w=1}$ has only one component, and so $E_{w=1}$ is a Hamilton circuit of $G$. ■

**Lemma 2.15** (Lemma 2.1 of [21]). *Let G be an edge-3-colorable cubic graph, w be a (1, 2)-eulerian weight of G, and $\mathcal{M}$ be a 1-factorization of G. If $E_{w=2} \notin \mathcal{M}$, then $(G, w)$ is non-Hamilton coverable.*

**Lemma 2.16.** *If a* 3*-connected cubic graph $G$ admits a Hamilton weight $w$ and the subgraph of $G$ induced by edges of $E_{w=1}$ is a Hamilton circuit, then $G$ is not bipartite.*

***Proof.*** We argue by contradiction and assume that $G$ is bipartite. Let $C = v_1 \cdots v_{2n} v_1$ be the Hamilton circuit induced by $E_{w=1}$. Then, for each $e = v_i v_j \in E_{w=2}$, $i$ and $j$ must have different parity since $G$ has no circuit of odd lengths. Let $v_1 v_{2k} \in E_{w=2}$. Then

$$M = \{v_i v_{i+1} : i \text{ is odd and } 2k+1 \leq i \leq 2n-1\} \cup \{v_1 v_{2k}\} \\ \cup \{v_j v_{j+1} : i \text{ is even and } 2 \leq j \leq 2k-2\}$$

is a perfect matching of $G$. Since $G$ is bipartite, $E(G) \backslash M$ is a 2-factor with each component as an even length circuit. So, $G$ has a 1-factorization $\mathcal{M} = \{M, M', M''\}$ containing $M$ as a member. Note that $E_{w=2} \cap M \neq \emptyset$ and $E_{w=2} \backslash M \neq \emptyset$. Thus $E_{w=2} \notin \mathcal{M}$, contrary to Lemma 2.15. ∎

For a Hamilton circuit $C = v_1 \cdots v_{2n} v_1$ of a cubic graph $G$, a chord $v_i v_j$ of $C$ is of *pace* $h$ where $h$ is the minimum of $|j-i|$ and $2n - |j-i|$. Thus, by Lemma 2.16, we have the following lemma.

**Lemma 2.17.** *If a* 3*-connected cubic graph $G$ admits a Hamilton weight $w$ and the subgraph of $G$ induced by edges of $E_{w=1}$ is a Hamilton circuit $C = v_1 \cdots v_{2n} v_1$, then some chord $e \in E_{w=2}$ of $C$ must be of even pace.*

**Lemma 2.18.** *Let $G$ be a* 3*-connected cubic graph containing no Petersen minor and admitting a Hamilton weight $w$. If the subgraph of $G$ induced by edges of $E_{w=1}$ is a Hamilton circuit, then, for a chord $e \in E_{w=2}$ of $C$ with even pace, $(G \setminus \{e\}, w)$ has a faithful circuit cover and, furthermore, each faithful circuit cover of $(G \setminus \{e\}, w)$ must be a circuit chain of even length.*

***Proof.*** Let $C$ be the Hamilton circuit of $G$ induced by $E_{w=1}$. First we prove that $(G \setminus \{e\}, w)$ has a faithful circuit cover. Since the Hamilton circuit $C$ is still a Hamilton circuit in the underlying cubic graph $G'$ of $G \setminus \{e\}$, it is obvious that $G'$ is edge-3-colorable. By Lemma 2.8, $(G', w)$ has a faithful circuit cover.

By Lemma 2.12, each faithful circuit cover of $(G \setminus \{e\}, w)$ must be a circuit chain joining the endvertices of $e$. By Lemma 2.13 and by the assumption that $e$ has even pace, such a circuit chain must have even length. ∎

## 3. NON-HAMILTON COVERABLE GRAPHS

Let $\mathcal{F}$ be a family of subgraphs of a graph $G$. The weight $w_{\mathcal{F}} : E(G) \mapsto \{0, 1, 2, \cdots\cdots\}$ with $w_{\mathcal{F}}(e)$ equal to the number of members of $\mathcal{F}$ containing $e$, for each edge $e \in E(G)$, is called the *weight of $G$ induced by $\mathcal{F}$*.

The main result of this section, Theorem 3.5, plays a key role in Section 4, and is believed to have further applications in the studies of some related problems. We start with some preparations.

**Definition 3.1.** *Let G be a cubic graph, w be a* (1, 2)*-eulerian weight of G, and* $e_1$ *and* $e_2 \in E_{w=1}$*. The graph* $H(G; e_1, e_2)$ *is obtained from G by inserting a new vertex, say* $x_i$*, into each* $e_i \in \{e_1, e_2\}$ *and adding a new edge* $e^*$ *joining* $x_1$ *and* $x_2$*. The weight of the new graph* $H(G; e_1, e_2)$ *(also denoted by w, for convenience) coincides with the original weight with* $w(e^*) = 2$*.*

**Definition 3.2.** *Let G be a cubic graph, and* $e_1, e_2 \in E(G)$*. The edges* $e_1$ *and* $e_2$ *are eventually adjacent if there exists a sequence of* $\triangle \leftrightarrow Y$*-operations after which* $e_1$ *and* $e_2$ *become adjacent.*

**Lemma 3.3.** *Let* $e_1, e_2$ *be two distinct edges of a graph* $G \in \langle \mathcal{K}_4 \rangle \backslash \{K_2^3, K_4\}$*. Then the edges* $e_1$ *and* $e_2$ *are not eventually adjacent to each other if one of the followings happens.*

(1) $e_1$ *and* $e_2$ *are contained in the same member of the unique* 1*-factorization of G;*
(2) $e_1$ *and* $e_2$ *are contained in two distinct triangles of G;*
(3) $e_1$ *is contained in a triangle of G and* $e_1$ *is not adjacent to* $e_2$*.*

***Proof.*** (1) is obvious. (2) is a corollary of (2) of Lemma 2.6. The following is the proof for (3).

We prove by induction on $|V(G)|$. It is obvious for $|V(G)| = 6$. Let $e_1 = u_1u_2$ be contained in a triangle $C_1 = u_1u_2u_3u_1$ and, by (1), we assume that $e_i \in M_i$ for $i = 1, 2$ where $\{M_1, M_2, M_3\}$ is the 1-factorization of $G$. Let $v_i \notin V(C_1)$ $(i = 1, 2, 3)$ be the vertex of $G$ adjacent to $u_i$. Since $e_2$ is not adjacent to $e_1 = u_1u_2$ and $e_2 \notin M_1$, $e_2 \notin E(C_1) \cup \{u_1v_1, u_2v_2, u_3v_3\}$. By (1) of Lemma 2.6, let $C_2$ be a triangle of $G$ distinct from $C_1$. By (2) above, we may assume that $e_2 \notin E(C_2)$. Let $G'$ be the graph obtained from $G$ by contracting $C_2$. By (2) of Lemma 2.6, no edge of $E(C_1) \cup \{u_1v_1, u_2v_2, u_3v_3, e_2\}$is contained in $C_2$. Thus, $e_2$ remains disjoint from $e_1$ and, $C_1$ remains as a triangle in $G'$. The lemma is proved by induction. ∎

**Lemma 3.4.** *Let* $G \in \langle \mathcal{K}_4 \rangle$ *and* $e_1, e_2 \in E(G)$*. Then* $H(G; e_1, e_2) \in \langle \mathcal{K}_4 \rangle$ *if and only if* $e_1$ *and* $e_2$ *are eventually adjacent.*

***Proof.*** "$\Leftarrow$" follows from Lemma 2.4 and from the definition of $\triangle \leftrightarrow Y$.

"$\Rightarrow$": Let $G$ be a smallest counterexample to the lemma. It is obvious that $G \neq K_2^3$. Thus, $H(G; e_1, e_2) \in \langle \mathcal{K}_4 \rangle \backslash \{K_2^3, K_4\}$. Let $x_1, x_2$ be the two new vertices introduced by subdividing $e_1$ and $e_2$ when defining $H(G; e_1, e_2)$. If $H(G; e_1, e_2)$ has a triangle $T$ containing neither $x_1$ nor $x_2$, $T$ is also a triangle of $G$. Applying a $\triangle \rightarrow Y$-operation by contracting $T$ to a vertex in $G$ and $H(G; e_1, e_2)$, we obtain new graphs $G'$ and $H'$ respectively, where $H' = H(G'; e_1, e_2)$. Since $H' \in \langle \mathcal{K}_4 \rangle$

and since $G$ is a smallest counterexample to the lemma, the edges $e_1, e_2$ are eventually adjacent in $G'$, so are in $G$. Thus, we may assume that *every triangle of* $H(G; e_1, e_2)$ *contains either* $x_1$ *or* $x_2$. Hence $H(G; e_1, e_2)$ has precisely two disjoint triangles, and so by Lemma 2.6, each of which contains exactly one of $x_1$ and $x_2$. It follows that these two triangles are 2-circuits in $G$, contrary to the fact that no graph in $\langle \mathcal{K}_4 \rangle \backslash \{K_2^3\}$ has a 2-circuit. ■

**Theorem 3.5.** *Let* $G \in \langle \mathcal{K}_4 \rangle$, $w : E(G) \mapsto \{1, 2\}$ *be a Hamilton weight of* $G$, *and* $e_1$ *and* $e_2 \in E_{w=1}$. *Then* $(H(G; e_1, e_2), w)$ *is non-Hamilton coverable if and only if* $e_1$ *and* $e_2$ *are not eventually adjacent.*

Let $G$, $w$ and $\{e_1, e_2\}$ be defined as in Theorem 3.5. We shall prove Lemma 3.6 for the necessity and Lemma 3.8 for the sufficiency of Theorem 3.5.

**Lemma 3.6.** *If* $(H(G; e_1, e_2), w)$ *is non-Hamilton coverable, then* $e_1$ *and* $e_2$ *are not eventually adjacent.*

***Proof.*** We argue by contradiction and assume that $e_1$ and $e_2$ are eventually adjacent. By Lemma 3.4, $H = H(G; e_1, e_2) \in \langle \mathcal{K}_4 \rangle$. Let $M = E_{w=2}(G)$ and $M' = E_{w=2}(H)$. Then $M' = M \cup \{e^*\}$, where $e^*$ is defined in Definition 3.1. Let $T$ be a 3-cut of $H$. If $e^* \notin T$, then $T$ can be viewed as a 3-cut of $G$, and so by Lemma 2.3, $|T \cap M'| = |T \cap M| = 1$. If $e^* \in T$, then since $G$ is 3-edge-connected, $T - e^*$ cannot be an edge-cut of $G$. Therefore, $T$ must consist of the three edges incident with $x_1$ or with $x_2$, and so $|T \cap M'| = 1$ also. By Lemma 2.3, $w$ is a Hamilton weight of $H(G; e_1, e_2)$, contrary to the assumption that $(H(G; e_1, e_2), w)$ is non-Hamilton coverable. ■

**Definition 3.7.** *Let* $w$ *be a* (1,2)*-eulerian weight of a cubic graph* $G$. *A* 1*-factorization* $\mathcal{M}$ *of* $G$ *is* $w$-matched *if* $E_{w=2} \in \mathcal{M}$, *and is* $w$*-unmatched if* $E_{w=2} \notin \mathcal{M}$.

By Lemma 2.15, if a cubic graph $G$ with a (1,2)-eulerian weight $w$ has a $w$-unmatched 1-factorization, then $(G, w)$ is non-Hamilton coverable. This is why we are to study $w$-unmatched 1-factorization.

**Lemma 3.8.** *Let* $G$, $w$, *and* $\{e_1, e_2\}$ *be defined as in Theorem* 3.5. *If the edges* $e_1$ *and* $e_2$ *are not eventually adjacent, then* $H(G; e_1, e_2)$ *has a* $w$*-unmatched* 1*-factorization.*

We shall prove Lemma 3.8 by induction. The following lemma and notions will be needed to reduce the order of the graph.

**Lemma 3.9.** *Let* $G$ *be a cubic graph and* $w$ *be a* (1,2)*-eulerian weight of* $G$ *such that* $E_{w=2}$ *is a perfect matching of* $G$, *and let* $e_1$ *and* $e_2 \in E_{w=1}$. *Assume that* $G$ *has a triangle* $C = abca$ *which contains* $e_2 = ab$ *but not* $e_1$ *and* $w(ac) = 2$. *Let* $e_2' = cc'$ *be the edge incident with* $c$ *but not in the triangle* $C$. *Let* $G'$ *be the graph obtained from* $G$ *by contracting* $C$ *to a vertex and* $w'$ *be the restriction of* $w$ *on* $G'$. *(See Fig.* 2*). Then* $H(G; e_1, e_2)$ *has a* $w$*-unmatched* 1*-factorization if* $H(G'; e_1, e_2')$

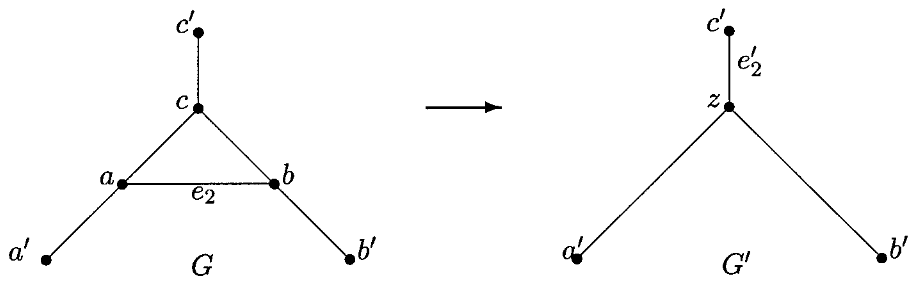


FIGURE 2. Operation defined in Lemma 3.9.

*has a $w'$-unmatched 1-factorization. (Note that $e_2$ and $e'_2$ are contained in the same member of any 1-factorization of $G$).*

***Proof.*** Let $H = H(G; e_1, e_2)$. The new vertex created by the contraction is denoted by $z$, and the vertex inserted into $e'_2$ is denoted by $x'_2$. Let $\mathcal{M}' = \{M'_1, M'_2, M'_3\}$ be a $w'$-unmatched 1-factorization of $H(G', e_1, e'_2)$. Without loss of generality and by symmetry, we may assume that $zx'_2 \in M'_1$ and $zb' \in M'_3$. Then $za' \in M'_2$, and either $x'_2x_1 \in M'_2$ or $x'_2x_1 \in M'_3$. (Note that $x_1$ and $x'_2$ are vertices of $H(G', e_1, e'_2)$ inserted into edges $e_1$ and $e'_2$, respectively).

**Case 1.** $x'_2x_1 \in M'_3$. *Then* $x'_2c' \in M'_2$.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 1-factorization of $H$ such that

$$\begin{aligned} M_1 &= [M'_1 \backslash \{zx'_2\}] \cup \{x_2a, cb\}, \\ M_2 &= [M'_2 \backslash \{za', x'_2c'\}] \cup \{c'c, a'a, x_2b\}, \\ M_3 &= [M'_3 \backslash \{zb', x'_2x_1\}] \cup \{x_1x_2, b'b, ca\}. \end{aligned}$$

Since $\mathcal{M}'$ is $w$-unmatched, $(M'_1 \cap E_{w'=2}(G')) \cup (M'_2 \cap E_{w'=2}(G')) \neq \emptyset$. It follows that $(M_1 \cap E_{w=2}(H)) \cup (M_2 \cap E_{w=2}(H)) \neq \emptyset$ also, and so $\mathcal{M}$ is $w$-unmatched.

**Case 2.** $x'_2x_1 \in M'_2$. *Then* $x'_2c' \in M'_3$.

Let $\mathcal{M} = \{M_1, M_2, M_3\}$ be a 1-factorization of $H$ such that

$$\begin{aligned} M_1 &= [M'_1 \backslash \{zx'_2\}] \cup \{ac, bx_2\}, \\ M_2 &= [M'_2 \backslash \{za', x'_2x_1\}] \cup \{a'a, cb, x_1x_2\}, \\ M_3 &= [M'_3 \backslash \{zb', x'_2c'\}] \cup \{c'c, b'b, x_2a\}. \end{aligned}$$

Obviously $\mathcal{M}$ is $w$-unmatched since $\mathcal{M}'$ is $w'$-unmatched. ■

***Proof of Lemma 3.8.*** Assume that $e_1$ and $e_2$ are not eventually adjacent and we shall argue by induction. Let $H = H(G; e_1, e_2)$.

The lemma is obvious for graphs with at most four vertices: Any two edges in $K_2^3$ are adjacent; and if $e_1, e_2 \in E(K_4)$ are not adjacent, then $H(K_4; e_1, e_2) = K_{3,3}$, which has a $w$-unmatched 1-factorization. If $G$ has a triangle $C_0$ such that $E(C_0) \cap \{e_1, e_2\} = \emptyset$, then obtain $G_1 \in \langle \mathcal{K}_4 \rangle$ from $G$ by contracting $C_0$ into a single vertex, and let $w_1$ denote the restriction of $w$ to $E(G_1) = E(G) \backslash E(C_0)$. By induction, $H(G_1; e_1, e_2)$ has a $w_1$-unmatched 1-factorization $\mathcal{M}_1$. It is straight forward to check that $\mathcal{M}_1$ would induce a $w$-unmatched 1-factorization of $H(G; e_1, e_2)$.

Therefore every triangle of $G$ must contain exactly one edge of $\{e_1, e_2\}$, and so $G$ has exactly two triangles $C_1$ and $C_2$ where $e_i \in E(C_i)$ for $i = 1, 2$. By Lemma 2.6, $G \cong \Gamma(n, \phi)$ for some $n$ and $\phi$ (see Definition 2.5). Assume that $C_1 = x_0y_0z_0x_0$, $C_2 = x_0'y_0'z_0'x_0'$ and $e_2 = y_0'z_0'$. Let $e_2'$ be the edge of $G \backslash E(C_2)$ incident with $x_0'$. By Lemma 3.3 (3), $e_2'$ and $e_1$ are eventually adjacent if and only if $e_2'$ and $e_1$ are adjacent in $G$ (and $G' = G/E(C_2)$ as well).

If $e_1$ and $e_2'$ are not adjacent (not eventually adjacent, either) in $G' = G/E(C_2)$, then, by induction, $H(G', e_1, e_2')$ has a $w$-unmatched 1-factorization. By Lemma 3.9, $H(G, e_1, e_2)$ has a $w$-unmatched 1-factorization as well.

So, assume that $e_1$ and $e_2'$ are adjacent in $G$. In this case, $G \cong \Lambda(m)$ where $m = n + 3$, by Lemma 2.6 (4). We shall use the notations in Definition 2.5, for $\Lambda(m)$, and so $V(\Lambda(m)) = \{v_1, v_2, \ldots, v_{2m}\}$ and $e_2' = v_1v_{m+1}$ (see Fig. 3). Thus, $e_2 = v_2v_{2m}$, $e_1$ is either $v_mv_{m+1}$ or $v_{m+1}v_{m+2}$ and $C_1 = v_1v_2v_{2m}v_1$ and $C_2 = v_mv_{m+1}v_{m+2}v_m$. Without loss of generality, we may assume that $w(v_1v_2) = 2$. Thus, $E_{w=2} = \{v_{2i-1}v_{2i} | i = 1, \ldots, m\}$. Furthermore $e_1 = v_mv_{m+1}$ if $m$ is even, and $e_1 = v_{m+1}v_{m+2}$ if $m$ is odd since $w(e_1) = 1$ and $e_1$ is adjacent to $e_2' = v_1v_{m+1}$.

Thus we shall define $M_1$ as follows.

If $m$ is even, then let

$$M_1 = \{e^*, v_1v_{m+1}, v_mv_{m+2}\} \cup \left\{ v_{2i}v_{2i+1} \mid 1 \le i \le \frac{m-2}{2} \right\} \cup \left\{ v_{2j-1}v_{2j} \mid \frac{m+4}{2} \le j \le m \right\}.$$

If $m$ is odd, then let

$$M_1 = \{e^*, v_1v_{m+1}\} \cup \left\{ v_{2i}v_{2i+1} \mid 1 \le i \le \frac{m-1}{2} \right\} \cup \left\{ v_{2j-1}v_{2j} \mid \frac{m+3}{2} \le j \le m \right\}.$$

In either case, $H(\Lambda(m); e_1, e_2) \backslash M_1$ is a disjoint union of even circuits, and $M_1$ contains some edges in $E_{w=2}$ (such as, $e^*$, $v_{2m-1}v_{2m}$, etc.) and some edges in $E_{w=1}$ (such as, $v_1v_{m+1}$, $v_2v_3$, etc.). Therefore $H(G; e_1, e_2)$ has a $w$-unmatched 1-factorization. ■

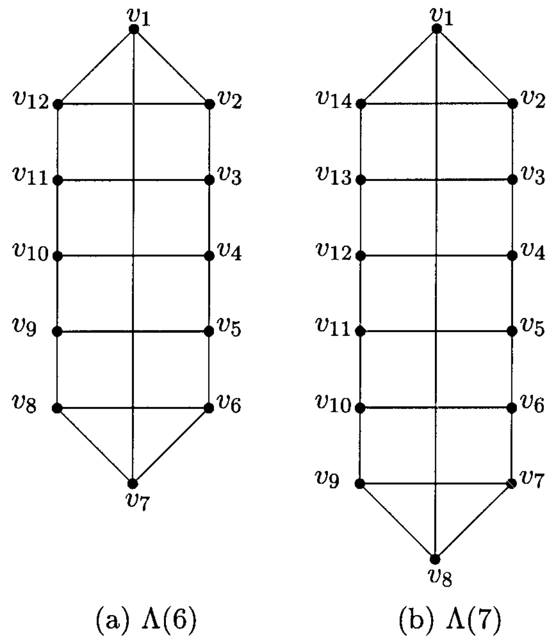


FIGURE 3. (a) $\Lambda(6)$, (b) $\Lambda(7)$.

## 4. THE PROOFS OF THE MAIN RESULTS

The sufficiency of Theorem 1.1 follows trivially from Lemma 2.3. We shall argue by contradiction to prove the necessity. Choose $(G, w)$ to be a counterexample to Theorem 1.1 with $|V(G)|$ minimized. That is, $w$ is a Hamilton weight of $G$ but $G \notin \langle \mathcal{K}_4 \rangle$.

**I.** By the minimality of $G$ and by Lemma 2.7, $G$ is 3-connected and cyclically 4-edge-connected.

In fact, if $G = G_1 \circ_{(v_1,v_2)} G_2$ for two cubic graphs $G_1$ and $G_2$, then $w_1$ and $w_2$ are Hamilton weights of $G_1$ and $G_2$, respectively. By the minimality of $G$, $G_1, G_2 \in \langle \mathcal{K}_4 \rangle$, and so by Lemma 2.7, $G \in \langle \mathcal{K}_4 \rangle$ also, a contradiction.

**II.** By Lemma 2.14, the subgraph of $G$ induced by $E_{w=1}$ is a Hamilton circuit $C$ of $G$. By Lemmas 2.17 and 2.18, it follows that

*$G$ has an edge $e \in E_{w=2}$ which is an even pace chord of $C$; and $(G \backslash \{e\}, w)$ has a faithful circuit cover $\mathcal{C} = \{C_1, \ldots, C_{2t}\}$ with $|\mathcal{C}| = 2t$ maximized, where $\mathcal{C}$ is a circuit chain joining the end vertices $x$ and $y$ of $e$. That is, $x \in C_1$, $y \in C_{2t}$ and $C_i \cap C_j \neq \emptyset$ if and only if $i - j = \pm 1$.*

**III.** Let $J_i = C_i \cup C_{i+1}$ $(i = 1, \ldots, t-1)$. Let $w_i$ be the weight of $J_i$ induced by the family of circuits $\{C_i, C_{i+1}\}$. For a technical reason, let $\{x\} = C_0$ and $\{y\} = C_{2t+1}$. We claim that

*the underlying graph $\overline{J_i} \in \langle \mathcal{K}_4 \rangle$, for each $i \in \{1, \ldots, 2t-1\}$.*

Since $G$ is a smallest counterexample to the theorem, it suffices to show

$\forall i \in \{1, \ldots, 2t-1\}$, *$\overline{J_i}$ is 3-connected, admits a Hamilton weight and has no Petersen minor.*

By the maximality of $\mathcal{C}$, the induced weight $w_i$ is a Hamilton weight for each $\overline{J_i}$. As each $J_i$ is a subgraph of $G$, $J_i$ contains no subdivision of the Petersen graph. It remains to show that $\overline{J_i}$ is 3-connected. Since $\overline{J_i}$ is cubic, it suffices to show that $\overline{J_i}$ has no 2-edge-cuts.

By contradiction, suppose that $T$ is a 2-edge-cut of $\overline{J_i}$ for some $i$. Let $R_1$ and $R_2$ be two components of $\overline{J_i} \setminus T$and let $\{D_1, \ldots, D_s\}$ be a circuit decomposition of $E_{w_i=1}$ where, by Lemma 2.10 (2), each $D_\mu$ is contained in either $R_1$ or $R_2$. If $C_{i-1} \cup C_{i+2}$ intersects with only one of $\{D_1, \ldots, D_s\}$, then $T$ is a 2-edge-cut of $G$, contrary to Claim I. Thus, there are two distinct circuits, say $D_1$ and $D_2$, of $\{D_1, \ldots, D_s\}$ such that $D_1 \cap C_{i-1} \neq \emptyset$ and $D_2 \cap C_{i+2} \neq \emptyset$. By Lemma 2.10 again, $\{C_i \triangle D_1, C_{i+1} \triangle D_1\}$ is also a Hamilton cover of $(\overline{J_i}, w_i)$. Thus, $C_1, \ldots, C_{i-1}, C_{i+1} \triangle D_1, C_{i+2}, \ldots, C_t$ is a circuit chain of $G \setminus \{e_0\}$ joining the endvertices of $e_0$ with a circuit $C_i \triangle D_1 (\neq \emptyset)$ excluded because

$$\begin{aligned} &(C_{i+1} \triangle D_1) \cap C_{i-1} \supseteq D_1 \cap C_{i-1} \neq \emptyset, \\ \text{and} \quad &(C_{i+1} \triangle D_1) \cap C_{i+2} \supseteq D_2 \cap C_{i+2} \neq \emptyset. \end{aligned}$$

This contradicts Lemma 2.12. Thus, $\overline{J_i}$ must be 3-connected and so Claim III is proved.

For each $i$, call a subdivided edge $L$ of $J_i$ *an attachment of* $C_{i-1}$ (or $C_{i+2}$) if $C_{i-1} \cap L \neq \emptyset$ (or $C_{i+2} \cap L \neq \emptyset$, respectively).

**IV.** Suppose that some $\overline{J_i}$, where $1 \leq i \leq 2t$, has attachment edges $e_1 \in (E(J_i) \setminus E(C_{i+1}))$ of $C_{i-1}$ and $e_2 \in (E(J_i) - E(C_i))$ of $C_{i+2}$, respectively. Then we claim that $e_1$ and $e_2$ are eventually adjacent in $\overline{J_i}$.

By contradiction, assume that $e_1$ and $e_2$ are not eventually adjacent. By Claim III and by Theorem 3.5, let $D_1$ and $D_2$ be the circuits of a non-Hamilton cover $\mathcal{C}_i$ of $H(\overline{J_i}; e_1, e_2)$ containing the new edge $e^*$. Let $G'$ be the subgraph of $G$ induced by edges contained in all circuits of $\mathcal{C} \setminus \{C_i, C_{i+1}\}$, the paths $D_1 \setminus \{e^*\}$ and $D_2 \setminus \{e^*\}$, and the edge $e_0$. Let $w'$ be the weight of $G'$ induced by the family of subgraphs $[\mathcal{C} \setminus \{C_i, C_{i+1}\}] \cup \{D_1 \setminus \{e^*\}, D_2 \setminus \{e^*\}, \{e_0\}, \{e_0\}\}$. The new weight $w'$ is eulerian, and the new graph $G'$ is bridgeless and Petersen minor free. By Theorem 2.9, $(G', w')$ has a faithful circuit cover $\mathcal{F}'$. Therefore, $\mathcal{F}' \cup [\mathcal{C}_i \setminus \{D_1, D_2\}]$ is a faithful Hamilton cover of $(G, w)$. This contradicts to the assumption that $w$ is a Hamilton weight of $G$ since $\mathcal{C}_i \setminus \{D_1, D_2\} \neq \emptyset$. This proves Claim IV.

**V.** We claim that $t \geq 2$. That is, the circuit chain $\mathcal{C}$ is of length at least 4. Assume that $t = 1$. That is, the circuit chain $\mathcal{C} = \{C_1, C_2\}$ is of length two. By Claim III, $\overline{J_1} = \overline{C_1 \cup C_2} \in \langle \mathcal{K}_4 \rangle$. Let $e_1, e_2$ be the subdivided edges of $C_1 \cup C_2$ containing the endvertices of $e$. Thus, $G = H(\overline{C_1 \cup C_2}; e_1, e_2)$. By Theorem 3.5,

$e_1$ and $e_2$ must be eventually adjacent since $w$ is a Hamilton weight of $G$. By Lemma 3.4, $G = H(\overline{C_1 \cup C_2}; e_1, e_2) \in \langle \mathcal{K}_4 \rangle$, contrary to the assumption that $(G, w)$ is a counterexample, and so Claim V holds.

**VI.** If $C_{i+2}$ has only one attachment in $J_i$ when $i + 1 < 2t$ (or, if $C_{i-1}$ has only one attachment in $J_i$ when $i > 1$), then $G$ has a cyclical 3-edge-cut consisting of the edge $e_0$ and two edges of $C_{i+1}$ (or two edges of $C_i$, respectively), contrary to Claim I. Thus

*each of $C_{i-1}$ (when $i > 1$) and $C_{i+2}$ (when $i + 1 < 2t$) has at least two attachments in $J_i$.*

Fix an $i \in \{2, \dots, 2t - 2\}$. Assume that $\overline{J_i} \in \langle \mathcal{K}_4 \rangle \setminus \{K_2^3, K_4\}$. By Lemma 2.6 (1) and (2), let $Q_1, \dots, Q_q$ be disjoint triangles of $\overline{J_i}$. Since $G$ is cyclically 4-edge-connected (by I.), each $Q_\mu$ ($\mu = 1, \dots, q$) contains an attachment of either $C_{i-1}$ or $C_{i+2}$. If $Q_1$ contains an attachment $e_1$ of $C_{i-1}$ and $Q_2$ contains an attachment $e_2$ of $C_{i+2}$, then $e_1$ and $e_2$ are not eventually adjacent (by Lemma 3.3 (2)). This contradicts IV. Hence, we assume that every triangle $Q_\mu$ contains an attachment $e_\mu$ of $C_{i-1}$ and contains no attachment of $C_{i+2}$. By VI, $\overline{J_i}$ has at least two attachments $f_1, f_2, \dots, f_\alpha$ of $C_{i+2}$. Since every pair of $f_\nu, e_\mu$ are eventually adjacent (by IV), all of these $f_1, f_2, \dots$ must be adjacent to every $e_1, \dots, e_q$ (by Lemma 3.3). Note that none of $\{f_1, f_2, \dots, f_\alpha\}$ is contained in any triangle $Q_\mu$ ($\mu = 1, \dots, q$), and $\alpha = |\{f_1, f_2, \dots, f_\alpha\}| \geq 2$ (by VI). This contradicts that $\overline{J_i}$ is cubic.

Thus *each $\overline{J_i}$ is either $K_4$ or $K_2^3$ for each $i = 2, \dots, 2t - 2$.* However, by Claim VI, $\overline{J_i} \neq K_2^3$, for each $i = 1, \dots, 2t - 1$. Therefore, the next claim obtains.

**VII.** $\overline{J_i} = K_4$, for each i with $2 \leq i \leq 2t - 2$, and $\overline{J_i} \in \langle \mathcal{K}_4 \rangle \setminus \{K_2^3\}$ for each $i \in \{1, 2t - 1\}$.

**VIII.** For any $1 \leq i \leq 2t - 2$, if $\overline{J_i} = K_4$, then $\overline{J_{i+1}} \neq K_4$.

By contradiction and without loss of generality, we assume that both $\overline{J_i} = \overline{C_i \cup C_{i+1}} = K_4$ and $\overline{J_{i+1}} = \overline{C_{i+1} \cup C_{i+2}} = K_4$. Consider the subgraph $C_i \cup C_{i+1} \cup C_{i+2}$, depicted in Figure 4. Since $t \geq 2$, we may assume, without loss of generality, that $i + 2 \leq 2t - 1$. Thus $i + 3 \leq 2t$.

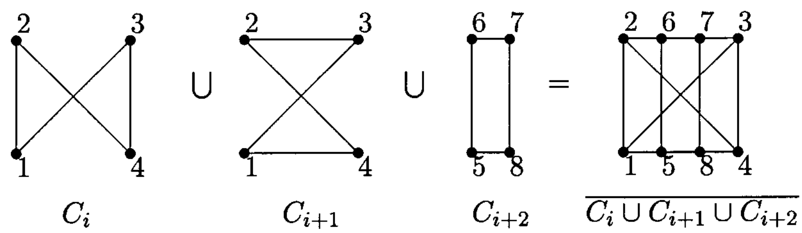


FIGURE 4. $\overline{C_i \cup C_{i+1} \cup C_{i+2}}$ where $\overline{C_i \cup C_{i+1}} \cong K_4$ and $\overline{C_{i+1} \cup C_{i+2}} \cong K_4$.

We shall use the notation in Figure 4 to proceed in the proof for Claim VIII. By Claim VI, 12 is an attachment of $C_{i-1}$ and both 56 and 78 are attachments of $C_{i+3}$. Note that the circuit 124856731 is adjacent to both $C_{i-1}$ and $C_{i+3}$. Thus, by replacing $C_i$, $C_{i+1}$, and $C_{i+2}$ of the circuit chain $\mathcal{C}$ with the circuits 124856731 and 134267851, we obtain a new faithful circuit cover of $(G\setminus\{e\})$ which is not a circuit chain joining $x$ and $y$, contrary to Lemma 2.12. This proves Claim VIII.

By Claim V, $t \geq 2$. If $t \geq 3$, then by Claim VII, both $\overline{J_2} = K_4$ and $\overline{J_3} = K_4$, contrary to Claim VIII. Therefore

$$t = 2,$$

and so by Claims VII and VIII, $(G\setminus\{e_0\})$ has a faithful cover $\{C_1, C_2, C_3, C_4\}$ such that $\overline{J_1}, \overline{J_3} \in \langle\mathcal{K}_4\rangle\setminus\{K_2^3, K_4\}$, and $\overline{J_2} \cong K_4$.

**IX.** $J_1$ has exactly two edges of attachments of $C_3$, since $\overline{J_2} = K_4$. It follows that $C_3 - E(C_2)$ consists of exactly two maximal subdivided edges in $J_2$.

**X.** By Claim I, $G$ is cyclically 4-edge-connected. Therefore each triangle of $\overline{J_1}$ must contain an edge of attachment of $C_0$ or of $C_3$.

**XI.** Let $e_0$ be the subdivided edge of $C_1$ containing $\{x\} = C_0$. By Claims IV and X and Lemma 3.3 (2), we conclude that $e_0$, the only edge of attachment of $C_0$ in $\overline{J_1}$ is not in any triangle of $\overline{J_1}$, that $\overline{J_1}$ has exactly two triangles $T_1$ and $T_2$, and that $\overline{J_1}$ has two edges $e_1, e_2$ of attachments of $C_3$ with $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$. By Claim IV and Lemma 3.3 (3), $e_0$ is adjacent to both $e_1$ and $e_2$. By Lemma 2.6 (4) and by the assumption that $\overline{J_1} \notin \{K_2^3, K_4\}$, $\overline{J_1} \cong \Lambda(n)$ for some $n \geq 0$ (Fig. 5).

We shall use the notations in Lemma 2.6 for $\Lambda(n)$, and so $V(\Lambda(n)) = \{v_1, v_2, \ldots, v_{2n}\}$. Without loss of generality, we assume that $T_1 = v_1v_2v_{2n}v_1$ and $T_2 = v_nv_{n+1}v_{n+2}v_n$, and that $C_1 = v_{n+1}v_1v_2v_{2n}v_{2n-1}v_3\cdots v_{n+2}v_{n+1}$ when $n$ is even or $C_1 = v_{n+1}v_1v_2v_{2n}v_{2n-1}v_3\cdots v_nv_{n+1}$ when $n$ is odd; $C_2 = v_1v_2v_3\cdots v_{2n}v_1$; and the vertex $x$ is contained in the subdivided edge $v_1v_{n+1}$.

By Claim VIII, $\overline{J_2} = K_4$ and by Claim IX, $J_1$ has two edges of attachment of $C_3$ which must be in distinct triangles of $\overline{J_1}$. Thus we may assume that these edges of attachment of $C_3$ in $J_1$ are the subdivided edges of $J_1$: $v_1bav_{2n}$ in $T_1$ with $ab \in E(C_2)\cap E(C_3)$; and $v_ncdv_{n+1}$ in $T_2$ (if $n$ is even), or $v_{n+1}cdv_{n+2}$ in $T_2$ (if $n$ is odd), with $cd \in E(C_2)\cap E(C_3)$. Since $\overline{J_2} = K_4$, the subdivided edges $ac$ and $bd$ are in $E(C_3)$, and so $C_2 \cup \{ac, bd\} = K_4$. Note that the two edges of attachment of $C_4$ in $J_2$ must then be the subdivided edges $ac$ and $bd$ of $C_3$.

Let $N = C_1 \cup C_2 \cup C_3$. We shall construct another faithful circuit cover $\{C_1', C_2', C_3'\}$ of $N$ with respect to the weight induced by $\{C_1, C_2, C_3\}$. We shall adopt the following notation in the construction: If $C = z_1z_2\cdots z_mz_1$ denotes a circuit, then $z_iCz_j$ denotes the section $z_iz_{i+1}\cdots z_{j-1}z_j$ of $C$.

If $n$ is even, then let

$$C_1' = v_1v_2v_3C_1v_{n+1}v_1,\ C_2' = bav_{2n}v_{2n-1}v_3C_2v_ncdb,\ C_3' = v_1bacdv_{n+1}C_2v_{2n}v_2v_1.$$

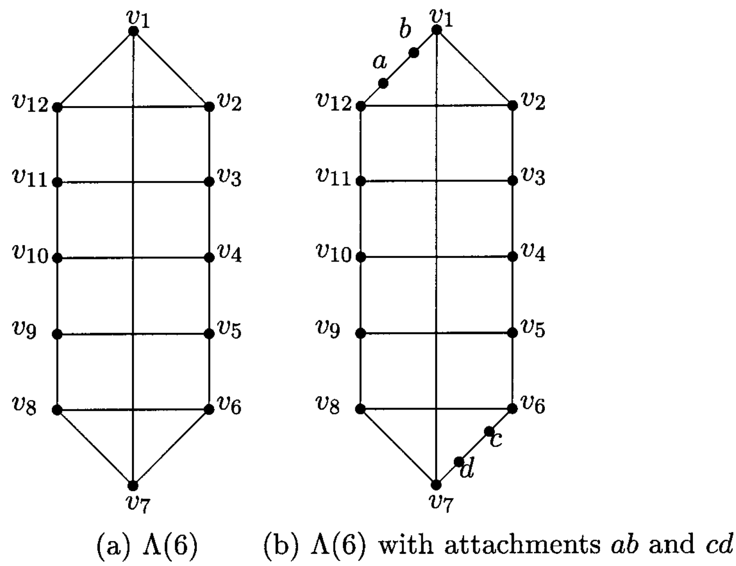


FIGURE 5. (a) $\Lambda(6)$. (b) $\Lambda(6)$ with attachments *ab* and *cd*.

If $n$ is odd, then let

$$C_1' = v_1v_2v_3C_1v_{n+1}v_1,\ C_2' = bav_{2n}v_{2n-1}v_3C_2v_{n+1}cdb,\ C_3' = bacdv_{n+2}C_2v_{2n}v_2v_1b.$$

In either case, $\{C_1', C_2', C_4\}$ forms a circuit chain joining the end vertices $x$ and $y$ of $e$ with $C_3'$ excluded, contrary to Lemma 2.12. This completes the proof. ■

In the proof of Corollary 1.2, we shall use the same notation as in Section 1 for the definition of $G_1 \circ_{(e_1,e_2)} G_2$. For each $i \in \{1,2\}$, let $(G_i, w_i)$ be a cubic graph with a (1,2)-weight $w_i$ and let $e_i \in E(G_i)$ be an edge with $w_i(e_i) = 2$. Let $G' = G_1 \circ_{(e_1,e_2)} G_2$ and let $w' : E(G') \mapsto \{1,2\}$ be such that $w'(e) = w_i(e)$ if $e \in E(G')\backslash\{e_x, e_y\}$, and $w'(e_x) = w'(e_y) = 2$. Part(i) of the following lemma follows from the facts that $\{e_x, e_y\} \subset E_{w'=2}(G')$ and that $e_i \in E_{w_i=2}(G_i)$, with $1 \le i \le 2$; and Part(ii) of it follows from the definition of $G'$ and from the fact that the Petersen graph is 3-connected.

**Lemma 4.1.** *With $G'$ defined as above, each of the following holds:*

(i) *$w'$ is a Hamilton weight of $G'$ if and only if both $w_1$ is a Hamilton weight of $G_1$ and $w_2$ is a Hamilton weight of $G_2$.*
(ii) *$G'$ does not have a Petersen minor if and only if neither $G_1$ nor $G_2$ has a Petersen minor.*

**Corollary 4.2.** *Let $(G', w')$ be a cubic graph with a Hamilton weight $w'$. Then either $G'$ is 3-connected, or there exist cubic graphs $(G_1, w_1)$ and $(G_2, w_2)$ with*

*Hamilton weights, such that, for some edges* $e_1 \in E(G_1)$ *and* $e_2 \in E(G_2)$ *with* $w_1(e_1) = w_2(e_2) = 2$, $G' = G_1 \circ_{(e_1,e_2)} G_2$.

***Proof.*** Since $G'$ admits a Hamilton weight, $G'$ is 2-connected. Suppose that $G'$ is not 3-connected. Then $G'$ has an edge 2-cut $T = \{e_x, e_y\}$ (say). By Lemma 2.10, $T \subset E_{w'=2}(G')$. Let $G'_1$ and $G'_2$ denote the two components of $G \setminus T$ and assume that $e_x = x_1x_2$ and $e_y = y_1y_2$, where $x_1, y_1 \in V(G'_1)$ and $x_2, y_2 \in V(G'_2)$. Since $G'$ is 2-connected and cubic, $x_1 \neq y_1$ and $x_2 \neq y_2$. Fix an $i \in \{1,2\}$. Let $G_i$ be the graph obtained from $G'_i$ by adding a new edge $e_i = x_iy_i$. Let $w_i\colon E(G_i) \mapsto \{1,2\}$ be such that $w_i(e) = w'(e)$ if $e \notin \{e_1, e_2\}$, and $w_1(e_1) = w_2(e_2) = 2$. Then $G' = G_1 \circ_{(e_1,e_2)} G_2$, and by Lemma 4.1, both$(G_1, w_1)$ and $(G_2, w_2)$ are cubic graphs with Hamilton weights. ■

We are now ready to prove Corollary 1.2. It suffices to prove the necessity.

Let $G$ be a cubic graph without a Petersen minor. If $G$ is 3-connected, then by Theorem 1.1, $G \in \langle \mathcal{K}_4 \rangle$. Assume that $G$ has an edge 2-cut. Then by Corollary 4.2, there exist cubic graphs $G_1$ and $G_2$, each of which admits a Hamilton weight, such that, for some edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, $G = G_1 \circ_{(e_1,e_2)} G_2$. Note that since both $G_1$ and $G_2$ are cubic graphs, $|V(G_i)| \leq |V(G)| - 2$ for each $i \in \{1,2\}$. Thus Corollary 1.2 obtains by applying induction to both $G_1$ and $G_2$. ■

## 5. CRITICAL CONTRA PAIRS AND PETERSEN MINORS

**Definition 5.1.** *A cubic graph G is called a permutation graph if G has a 2-factor F which is the union of two chordless circuits.*

**Definition 5.2.** *An ordered pair* $(G, w)$ *is called a contra pair if w is an admissible eulerian weight of a graph G and* $(G, w)$ *has no faithful cycle cover.*

Let $\mathcal{G}$ be the collection of all ordered pairs $(G, w)$ such that $w$ is an admissible $(1,2)$-eulerian weight of a graph $G$. Define a partial ordering $\leq_w$ in $\mathcal{G}$ that

$$(G_1, w_1) \leq_w (G_2, w_2)$$

if $G_2$ has a subgraph $H_2$ that is isomorphic to a subdivision $H_1$ of $G_1$, and $w_2(e) \geq w_1(e)$ for each $e \in H_2 = H_1$.

It is proved in [1] and [2] that every minimal (under the ordering $\leq_w$) contra pair must be a permutation graph $G$ with $E_{w=1}$ as its 2-factor which is the union of two chordless circuits, and $E_{w=2}$ as the set of edges joining the components of $E_{w=1}$. Then, by applying a theorem of Ellingham [3], it is proved that there exists a Petersen minor in such a permutation graph. The main result in this section proves that under a certain 'minimal' condition, the Petersen minors appear *"almost everywhere"* in such a permutation graph.

**Definition 5.3.** *Let $w$ be an admissible $(1,2)$-eulerian weight of a cubic graph $G$. Let $C$ be a circuit of $G$ and let $w_{\overline{C}}$ be the eulerian weight of $G$ such that*

$$w_{\overline{C}}(e) = \begin{cases} w(e) & \text{if } e \notin E(C), \\ w(e) - 1 & \text{if } e \in E(C). \end{cases}$$

*$(G, w)$ is a critical contra pair if for each circuit $C$ of $G$, the eulerian weight $w_{\overline{C}}$ is not admissible.*

Let $P_{10}$ be the Petersen graph, and $M$ be a perfect matching of $P_{10}$. Define a $(1,2)$-eulerian weight $w_{10}$ of $P_{10}$ as follows.

$$w_{10}(e) = \begin{cases} 2 & \text{if } e \in M, \\ 1 & \text{otherwise.} \end{cases}$$

(see Fig. 6). One can prove that $(P_{10}, w_{10})$ does not have a faithful cycle cover. And by Theorem 2.9, $(P_{10}, w_{10})$ is the smallest contra pair.

**Conjecture 5.4** (Jackson [12]). *Let $G$ be a cyclically 5-edge-connected cubic graph and $w$ be a $(1,2)$-eulerian weight of $G$. If $(G, w)$ has no faithful cover, then $G = P_{10}$.*

**Conjecture 5.5** (Goddyn [7], or see [8]). *$(P_{10}, w_{10})$ is the only 3-connected and cyclically 4-edge-connected critical contra pair.*

**Conjecture 5.6** (Fleischner and Jackson, Conjecture 12 in [4]). *Let $G$ be a permutation graph such that $M$ is a perfect matching of $G$ and $G \setminus M$ is the union of two chordless circuits $C_1$ and $C_2$. If $(G, w)$ is a critical contra pair with $E_{w=2} = M$ and $E_{w=1} = E(C_1) \cup E(C_2)$, then $(G, w) = (P_{10}, w_{10})$.*

The main result of this part is a byproduct of the proof of Theorem 1.1 and is an approach to the above conjectures.

**Theorem 5.7.** *Let $(G, w)$ be a critical contra pair described in Conjecture 5.6. Let $e \in E_{w=2} = M$.*

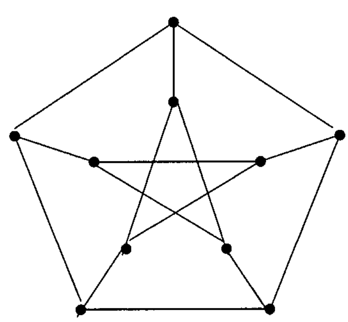

FIGURE 6. The Petersen graph $P_{10}$.

(A). *Then* $(G\backslash\{e\},w)$ *is faithful coverable. Furthermore, every faithful cover of* $(G\backslash\{e\},w)$ *is a circuit chain joining the endvertices of e.*
(B). *Choose a faithful cover* $\mathcal{F}=\{C_1,\ldots,C_s\}$ *of* $(G\backslash\{e\},w)$ *with* $|\mathcal{F}|=s$ *as large as possible. If* $s\geq 4$, *then* $\overline{C_i\cup C_{i+1}\cup C_{i+2}}$, *for each* $i\in\{1,\ldots,s-2\}$, *contains a Petersen minor.*

Actually, we are able to prove a slightly stronger result.

**Theorem 5.8.** *Let* $(G,w)$ *be a critical contra pair described in Conjecture* 5.6. *Let* $e\in E_{w=2}=M$.

(A). *Then* $(G\backslash\{e\},w)$ *is faithful coverable. Furthermore, every faithful cover of* $(G\backslash\{e\},w)$ *is a circuit chain joining the endvertices of e.*
(B). *Choose a faithful cover* $\mathcal{F}=\{C_1,\ldots,C_s\}$ *of* $(G\backslash\{e\},w)$ *with* $|\mathcal{F}|=s$ *as large as possible.*

(1) *Then* $s\geq 3$;
(2) *If* $\overline{C_i\cup C_{i+1}}$ *does not contain a Petersen minor, then* $\overline{C_i\cup C_{i+1}}\equiv K_4$ *for each* $i\in\{2,\ldots,s-2\}$, *and* $\overline{C_i\cup C_{i+1}}\in\langle\mathcal{K}_4\rangle\backslash\{K_2^3\}$ *for each* $i\in\{1,s-1\}$.
(3) *If* $s\geq 4$, *then, for each* $i\in\{2,\ldots,s-1\}$, *either* $\overline{C_{i-1}\cup C_i}$ *or* $\overline{C_{i+1}\cup C_i}$ *contains a Petersen minor.*

**Lemma 5.9.** *Let G be a permutation graph. Then G is cyclically* 4*-edge-connected if* $|V(G)|\geq 8$.

***Proof.*** Let $M$ be a perfect matching of $G$ such that $G\backslash M$ is the union of two chordless circuits $C_1$ and $C_2$. Since $C_1\cup C_2$ is a 2-factor of $G$, each cyclic edge-cut $T$ of $G$ must contain an even number of edges of $C_1\cup C_2$. Assume that a permutation graph $G$ is not cyclically 4-edge-connected. Let $T$ be a cyclic edge-cut of size at most 3. If $|T\cap(C_1\cup C_2)|=2$, then one of $\{C_1,C_2\}$, say $C_1$, does not intersect with one component of $G\backslash T$. Thus, some edge of $M$ must be a chord of $C_2$, and therefore, $G$ is not a permutation graph. If $|T\cap(C_1\cup C_2)|=0$, then $T\subseteq M$. It is not hard to see that $C_1$ (as well as $C_2$) is a spanning subgraph of a component of $G\backslash T$. Since each component of $G\backslash T$ is a chordless circuit, no component of $G\backslash T$ contains any edge of $M$. This implies that $M=T$ and therefore $|V(G)|=2|T|=2|M|\leq 6$. ■

***Proof of Theorem 5.8.*** Since $(G,w)$ is a contra pair, $G$ is not edge-3-colorable. Therefore, the length of the circuits $C_1$, $C_2$ must be odd and at least 5. Furthermore, for each edge $e\in M=E_{w=2}$, $\overline{G\backslash\{e\}}$ is edge 3-colorable: $M=E_{w=2}$ is colored red, edges of $C_1$ and $C_2$ in the underlying graph $\overline{G\backslash\{e\}}$ are alternately colored with blue and yellow. Hence, $(G\backslash\{e\},w)$ has a faithful circuit cover consisting of all red-blue alternately colored, and red-yellow alternately colored circuits.

For a faithful circuit cover $\mathcal{F}$ of $(G\backslash\{e\},w)$, let $\mathcal{C}\subseteq\mathcal{F}$ be a circuit chain of $G$ joining the endvertices of $e$. If $\mathcal{F}\backslash\mathcal{C}\neq\emptyset$, then for each $C\in\mathcal{F}\backslash\mathcal{C}$, $w_{\overline{C}}$ is admissible. This contradicts that $(G, w)$ is a critical contra pair. So, we have proved (A) that $\mathcal{F} = \mathcal{C}$ is a circuit chain.

The length of a circuit chain must be at least two since $G \setminus \{e\}$ is not a circuit. That is, $s \geq 2$. We next prove that $s \geq 3$ ((1) of Part (B)). Color the graph $\overline{G \setminus \{e\}}$ as described in the first paragraph. Swap the colors of $C_1$ if necessary so that the pair of subdivided edges of $G \setminus \{e\}$ (containing the endvertices of $e$) are colored the same, say, *blue*. Let $\mathcal{F}'$ be the faithful circuit cover induced by the coloring (described in the first paragraph). By (A), $\mathcal{F}' = \{C_1, \ldots, C_{s'}\}$ is a circuit chain. Since both $C_1$ and $C_{s'}$ contain endvertices of $e$, they are *red-blue* alternately colored. Hence, $s'$, the length of the chain $\mathcal{F}'$, must be an odd integer. Therefore, $s \geq s' \geq 3$.

By Lemma 5.9, $G$ is a cyclically 4-edge-connected permutation graph. Let $\mathcal{F} = \{C_1, \ldots, C_s\}$ be a faithful cover of $(G \setminus \{e_0\}, w)$ for some $e_0 = xy \in E_{w=2}$ with $|\mathcal{F}|$ as large as possible. Here $C_i \cap C_j \neq \emptyset$ if and only if $i = j \pm 1$ since $\mathcal{F}$ is a circuit chain. For a technical reason, let $\{x\} = C_0$ and $\{y\} = C_{s+1}$. Let $J_i = C_i \cup C_{i+1}$ $(i = 1, \ldots, s-1)$. Let $w_i = w_{\{C_1, C_{i+1}\}}$ be the weight of $J_i$ induced by the family of circuits $\{C_i, C_{i+1}\}$.

Since $|\mathcal{F}|$ is maximum, each $w_i = w_{\{C_i, C_{i+1}\}}$ is a Hamilton weight of $J_i$. By Theorem 1.1,

*either $J_i$ contains a Petersen minor or $J_i \in \langle \mathcal{K}_4 \rangle$.*

The rest of the proof is quite similar to that in the proof of Theorem 1.1, and so we only present an outline of it.

Similar to Claim VI in Section 4, we have that

*each of $C_{i-1}$ (when $i > 1$) and $C_{i+2}$ (when $i + 1 < s$) has at least two attachments in $J_i$. Therefore, $\overline{J_i} \neq K_2^3$ for each $i$.*

Similar to Claim VII in Section 4, we have that

*for each $i$ with $1 \leq i \leq s-1$, if $\overline{J_i}$ does not contain a Petersen minor, then $\overline{J_i} \equiv K_4$ for $2 \leq i \leq s-2$, and $\overline{J_i} \in \langle \mathcal{K}_4 \rangle \setminus \{K_2^3\}$ for each $i \in \{1, s-1\}$.*

This proves (2) of Theorem 5.8 (B).

When $s \geq 4$, similar to Claims VIII–XI in Section 4, we have that

*for each $i$ with $2 \leq i \leq s-2$, if $\overline{J_i} \equiv K_4$ then both $\overline{J_{i-1}}$ and $\overline{J_{i+1}}$ must contain a Petersen minor.*

This proves (3) of Theorem 5.8 (B). ∎

**Remark.** It was announced by Tom Fowler at the 28th Southeastern International Conference on Combinatorics, Graph Theory And Computing (Florida Atlantic University, Boca Raton, Florida, March, 1997), that Conjecture B of Fiorini and Wilson on unique edge-3-coloring for planar graphs has been solved by Robin Thomas and Tom Fowler with an approach similar to that for the 4-color theorem in [14].

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