(2 + ϵ)-Coloring of Planar Graphs with Large Odd-Girth

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Abstract: The odd-girth of a graph is the length of a shortest odd circuit. A conjecture by Pavol Hell about circular coloring is solved in this article by showing that there is a function $f(\epsilon)$ for each $\epsilon : 0 < \epsilon < 1$ such that, if the odd-girth of a planar graph G is at least $f(\epsilon)$, then G is $(2+\epsilon)$-colorable. Note that the function $f(\epsilon)$ is independent of the graph G and $\epsilon \to 0$ if and only if $f(\epsilon) \to \infty$. A key lemma, called the *folding lemma*, is proved that provides a reduction method, which maintains the odd-girth of planar graphs. This lemma is expected to have applications in related problems. © 2000 John Wiley & Sons, Inc. J Graph Theory 33: 109–119, 2000

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1. INTRODUCTION

Define the *odd-girth* of a graph to be the length of a shortest odd-length circuit in

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the graph. It is well known that every bipartite graph (a graph with odd-girth ∞) is 2-colorable and every triangle-free planar graph (a graph with odd-girth greater than or equal to five) is 3-colorable (Grötzsch [6]). In this article, we study planar graphs whose circular chromatic number (also known as *star chromatic number*) is between 2 and 3 under a condition that their odd-girths are sufficiently large.

The main result of this article is the following.

Theorem 1.1. *There is a function $f(\epsilon)$ for each $\epsilon > 0$ such that, if the odd-girth of a planar graph G is at least $f(\epsilon)$, then G is circular-$(2+\epsilon)$-colorable.*

Specifically, we show that, if the odd-girth of planar graph G is at least $10k-3$, then G is $\frac{2k+1}{k}$-colorable (see the definition of $\frac{p}{q}$-colorable below).

Theorem 1.1 was originally a conjecture proposed by Pavol Hell at the Southeastern Conference on Combinatorics, Graph Theory, and Computing at Boca Raton, Florida, in 1998.

Previous work on planar graphs whose circular chromatic number is between 2 and 3 was done by Moser [9], who showed that there exist planar graphs with circular chromatic number r, for every rational number r between 2 and 3. The circular chromatic number of series-parallel graphs with large girth was studied in [8], where it was shown that a series-parallel graph with girth at least $2\lfloor(3k-1/2)\rfloor$ has circular chromatic number at most $4k/(2k-1)$. In addition, Moser [9] and Zhu [14] have shown (using the Four Color Theorem) that K_5-minor free graphs have circular chromatic number between 2 and 4.

As a generalization of the conventional vertex coloring problem, the *circular coloring* problem, introduced in [12], is defined as follows.[1] Denote the color assigned to a vertex v as $c(v)$. A graph G has a circular-k-coloring with $k = \frac{p}{q}$ (p and q are positive integers), if there is a mapping $c : V(G) \mapsto \{0, 1, \ldots, p-1\}$ such that

$$p - q \geq |c(u) - c(v)| \geq q$$

for each $uv \in E(G)$. Extensive literature about the circular coloring problem can be found in a comprehensive survey article [15].

The following were proved in [12] (also see [3]).

(1) Let $k < h$. Then a graph G has a circular-h-coloring, if it has a circular-k-coloring;
(2) If k is an integer, then a graph G has a circular-k-coloring if and only if G has a k-coloring (a conventional vertex k-coloring).

Thus, without any confusion, we may simply say that "a graph G is k-colorable" if G is k-colorable whenever k is an integer, or if G is circular-k-colorable whenever k is a rational number.

[1] We adopt the new terminology of Hell and Zhu: *circular coloring* and *circular chromatic number* instead of the original terminology *star coloring* and *star chromatic number* introduced by Vince [12].

It is easy to check that, if a graph G contains an odd circuit of length less than $2k+1$, then the graph G cannot be r-colorable for any rational number $r \leq \frac{2k+1}{k}$. This is why the odd-girth of a $(2+\epsilon)$-colorable graph must be sufficiently large.

2. TERMINOLOGY AND NOTATION

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *walk*, W, of a graph G is a sequence of vertices $v_0 \cdots v_s$ such that $v_i v_{i+1}$ is an edge of G for each $i = 1, \ldots, s-1$. A walk may pass through a vertex or edge more than once. The *length* of a walk $W = v_0 \cdots v_s$ is s, the number of edges that W passes through, and is denoted by $l(W)$. A walk $W = v_0 \cdots v_s$ is *closed*, if $v_0 = v_s$. A walk (closed walk) $W = v_0 \cdots v_s$ that does not pass through any vertex more than once is called a *path* (*circuit*, respectively). A circuit of length one is called a *loop*. In this article, loops are not allowed.

Let $C = v_1 \cdots v_r v_1$ be a circuit. A *segment* of C is a path contained in C and is denoted by $v_i C v_j$, where v_i and v_j are the endvertices of the segment. Note that $v_i C v_j$ and $v_j C v_i$ $(i \neq j)$ are different; their union is the circuit C. For example, if $i < j$, then $v_i C v_j = v_i v_{i+1} \cdots v_{j-1} v_j$ and $v_j C v_i = v_j v_{j+1} \cdots v_r v_1 \cdots v_{i-1} v_i$ (mod r).

The *girth* of a graph G is the length of a shortest circuit of G. As stated above, the *odd-girth* of a graph is the length of a shortest odd circuit. A path $P = v_0 \cdots v_s$ is an *induced path*, if the degree of v_i is two for each $i = 1, \ldots, s-1$.

For a planar graph G embedded in the plane S, each connected topological region (an open set) of $S \backslash G$ is called *a face* of G. If the boundary of a face is a circuit of G, it is called a *facial circuit* of G.

For the purpose of this article, a graph (or subgraph) is called a *cycle*, if the degree of each vertex is even. For two subgraphs H_1 and H_2, the *symmetric difference* of $E(H_1)$ and $E(H_2)$, denoted by $E(H_1)\Delta E(H_2)$, is defined as $[E(H_1) \cup E(H_2)] \backslash [E(H_1) \cap E(H_2)]$. It is known that the symmetric difference of two cycles is a cycle.

3. MAIN RESULTS

The function $f(\epsilon)$ in Theorem 1.1 is described in Theorem 3.1.

Theorem 3.1. *Let G be a planar graph. If the odd-girth of G is at least* $10k-3$, *then G is* $\frac{2k+1}{k}$*-colorable.*

Nešetřil and Zhu [10], and independently, Galluccio, Goddyn, and Hell [5] proved that every planar graph with girth at least $10k-4$ is $\frac{2k+1}{k}$-colorable. This result is improved by Theorem 3.1 in that the girth requirement is relaxed with the odd-girth requirement, noting that $10k-4$ is even for all $k \geq 1$. With this improvement, the 2-coloring of bipartite (planar) graphs becomes an asymptotic conclusion of Theorem 3.1, since the odd-girth of a bipartite graph is ∞ (though its girth may be very small).

It is easy to show that Theorem 3.1 can also be presented as graph homomorphism result as follows.

Theorem 3.2. *Every planar graph with odd-girth at least* $10k-3$ *has a homomorphism to the circuit of length* $2k+1$.

4. FOLDING LEMMA FOR VERTEX IDENTIFICATION

The following *folding* lemma plays a key role in the proof of Theorem 3.1 and is believed to have further applications in related problems.

Lemma 4.1. *Let G be a planar graph with odd-girth g. If* $C = v_0 \cdots v_{r-1}v_0$ *is a facial circuit of G with* $r \neq g$, *then there is an integer* $i \in \{0, \ldots, r-1\}$ *such that the graph* G' *obtained from G by identifying* v_{i-1} *and* v_{i+1} (*mod* r) *is still of odd-girth* g.

The following lemmas are used in the proof of Lemma 4.1.

Lemma 4.2. *If a cycle contains an odd number of edges, it must contain an odd circuit.*

Proof. It is obvious that a cycle is the union of edge-disjoint circuits. ∎

Lemma 4.3. *Let G be a graph and W be a closed walk of G. If the length of W is odd, then the subgraph of G induced by edges of W contains an odd circuit.*

Proof. Let G' be the graph obtained from W (with the same vertex set $V(W)$) by replacing each edge $e \in W$ with t parallel edges, if the walk W passes through the edge e t times. Since W is a closed walk of odd length, the new graph G' is a cycle with an odd number of edges. By Lemma 4.2, G' contains an odd circuit C. ∎

Proof of Lemma 4.1. Suppose that $C = v_0 \cdots v_{r-1}v_0$ is a facial circuit of G with $r \neq g$. For each $i \in \{0, \ldots, r-1\}$, let G_i be the graph obtained from G by identifying v_{i-1} and v_{i+1} (*mod* r) of C. Assume that each $G_i, i \in \{0, \ldots, r-1\}$, is of odd-girth less than g (obviously, of odd-girth $g-2$). That is, for each $i \in \{0, \ldots, r-1\}$, G contains an odd circuit C_i of length g passing through the segment $v_{i-1}v_iv_{i+1}$ of C. This kind of circuit C_i is called a *critical circuit of G around C containing* $v_{i-1}v_iv_{i+1}$. Each critical circuit C_i of length g must contain a maximal segment $v_\mu C v_{\mu+p_i}$ of C with $v_{i-1}v_iv_{i+1} \subseteq v_\mu C v_{\mu+p_i}$, where p_i is called the *pace of* C_i *around* C.

Choose a critical circuit C_h with the largest pace p. That is, we assume that $v_\alpha C v_{\alpha+p}$ is a maximal segment of C contained in C_h. Consider another critical circuit C_α (as defined above, the critical circuit C_α contains the segment $v_{\alpha-1}v_\alpha v_{\alpha+1}$). Let $v_\beta C v_{\beta+q}$ $(v_{\alpha-1}v_\alpha v_{\alpha+1} \subseteq v_\beta C v_{\beta+q})$ be a maximal segment of C contained in C_α. By the choice of C_h, it is clear that $v_{\beta+q}$ is contained in the segment $v_{\alpha+1} C v_{\alpha+p-1}$ (*mod* r).

I. We claim that v_β *is contained in the segment* $v_{\alpha+p+1}Cv_{\alpha-1}$ (*mod* r). That is, $v_\beta, v_\alpha, v_{\beta+q}, v_{\alpha+p}$ are all distinct and are in this order around the facial circuit C. To the contrary, we suppose that v_β is not contained in the segment $v_{\alpha+p+1}Cv_{\alpha-1}$ (mod r). That is, $v_\alpha, v_{\beta+q}, v_\beta, v_{\alpha+p}$ are in this order around the facial circuit C and $v_\alpha, v_{\beta+q}, v_\beta$ are all distinct. Thus, the segment $v_\beta Cv_{\beta+q}$, which properly contains $v_{\alpha+p}Cv_\alpha$, is of length $q > r-p$. By the choice of C_h, which has the largest pace p among all critical circuits around C, we have that $p \geq q > r-p$. This implies that

$$p > \frac{r}{2}. \tag{1}$$

Case 1.1. r *is even*. By inequality (1), we need to prove only that the pace of any critical circuit C_i is at most $\frac{r}{2}$. To the contrary, suppose that the pace of a critical circuit C_i is greater than $\frac{r}{2}$. The symmetric difference $C^* = C_i \Delta C$ is a cycle. The cycle C^* contains an odd number of edges, since

$$|E(C^*)| = |E(C_i)| + |E(C)| - 2|E(C_i) \cap E(C)| \equiv |E(C_i)| + |E(C)| \equiv |E(C_i)|$$

(because C is of even length r in this case)

$$\equiv 1 \ (mod\ 2).$$

Therefore, by Lemma 4.2, the cycle C^* must contain an odd length circuit. But the cycle C^* does not have enough edges, since

$$|E(C^*)| = |E(C_i)| + |E(C)| - 2|E(C_i) \cap E(C)| < g + r - 2\left(\frac{r}{2}\right) = g.$$

This contradicts the fact that g is the odd-girth of G and, therefore, our assertion holds:

$$p \leq \frac{r}{2}.$$

This contradicts the inequality (1).

Case 1.2. r *is odd*. Since $r \neq g$, we must have that

$$r \geq g + 2.$$

We have that $v_\alpha, v_{\beta+q}, v_\beta, v_{\alpha+p}$ are in this order around the facial circuit C and $v_\alpha, v_{\beta+q}, v_\beta$ are all distinct. Let $Q_\alpha = C_\alpha \backslash \{v_{\beta+1}, v_{\beta+2}, \ldots, v_{\beta+q-1}\}$, which is a segment contained in C_α (deleting the segment $v_\beta Cv_{\beta+q}$ from the circuit C_α); let $Q = C \backslash \{v_{\beta+1}, v_{\beta+2}, \ldots, v_{\beta+q-1}\} = v_{\beta+q}Cv_\beta$. Since the circuit

$$C_\alpha = Q_\alpha \cup [v_\beta Cv_{\beta+q}]$$

is of length g, and the circuit

$$C = Q \cup [v_\beta Cv_{\beta+q}]$$

is of length r ($r > g$ and $r \equiv g$ (mod 2)), we have that

$$l(Q) \equiv l(Q_\alpha) \ (mod\ 2) \tag{2}$$

and

$$l(Q) > l(Q_\alpha). \tag{3}$$

Let $Q_h = C_h \backslash \{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_{\alpha+p-1}\}$, which is a segment contained in C_h (deleting the segment $v_\alpha C v_{\alpha+q}$ from C_h). Then

$$C_h = Q_h \cup [v_\alpha C v_{\alpha+p}] = Q_h \cup [v_\alpha C v_{\beta+q}] \cup Q \cup [v_\beta C v_{\alpha+p}].$$

Replacing Q with Q_α, we have a closed walk

$$W = Q_h \cup [v_\alpha C v_{\beta+q}] \cup Q_\alpha \cup [v_\beta C v_{\alpha+p}].$$

By (2) and (3), we have that

$$l(W) < l(C_h) = g$$

and

$$l(W) \equiv l(C_h) \equiv 1\ (mod\ 2).$$

By Lemma 4.3, the closed walk W contains a circuit of odd length less than g. This contradicts the fact that g is the odd-girth of G.

II. For the sake of convenience, denote $C_h = D''$, and denote the maximal segment $v_\alpha C v_{\alpha+p}$ of C contained in D'' by $v_b C v_d$; denote $C_\alpha = D'$ and denote the maximal segment $v_\beta C v_{\beta+q}$ of C contained in D' by $v_a C v_c$, where, by I,

$$0 \le a < b < c < d \le r - 1,$$

and

$$d - b = p$$

is the maximum pace among all critical circuits. Note that every critical circuit does not intersect with *interior* (C), since C is a facial circuit. The critical circuits D' and D'' must cross each other in *exterior* (C), since G is a planar graph. Let w be a vertex in the intersection of $D' \backslash C$ and $D'' \backslash C$.

Assign an orientation around C such that v_a, v_b, v_c, v_d are in this order. Denote the (oriented) segments of C from v_a to v_b by P_1, v_b to v_c by P_2, v_c to v_d by P_3, each of which is of positive length, since a, b, c, and d are distinct. Assign orientations around D' and D'' such that v_a, v_b, v_c are in this order in D', and v_b, v_c, v_d are in this order in D''. Denote the (oriented) segments of D' from v_c to w by Q_3, w to v_a by Q_1. That is, the oriented circuit D' is $P_1P_2Q_3Q_1$. Denote the (oriented) segments of D'' from v_d to w by Q_4, w to v_b by Q_2. That is, the oriented circuit D'' is $P_2P_3Q_4Q_2$. (See Fig. 1.)

For an oriented path P, the path with the opposite direction of P is denoted by $\tilde{P}$. For a walk W, note that (a) $l(W)$ might be greater than the number of edges contained in W, since some edges might be traversed by the walk more than once; (b) if a walk W is path or a circuit, $l(W) = |E(W)|$.

Here, we have a few closed walks obtained by combinations of those segments:

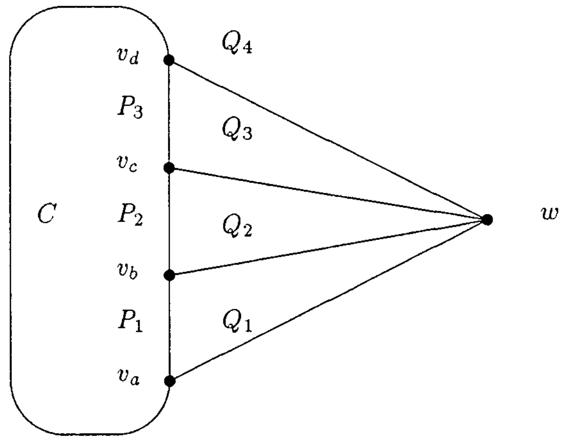

FIGURE 1. Two critical circuits D' and D'' around the facial circuit C.

(1) $W_1 = P_1\tilde{Q}_2Q_1$,
(2) $W_2 = P_2Q_3Q_2$,
(3) $W_3 = P_3Q_4\tilde{Q}_3$,
(4) $W_0 = P_1P_2P_3Q_4Q_1$.

Note that $l(W_1) + l(W_2) = l(D') + 2l(Q_2)$. Thus,

$$l(W_1) + l(W_2) \equiv l(D') \equiv 1 \ (mod\ 2). \tag{4}$$

Similarly, by comparing W_2 and W_3, we have that

$$l(W_3) + l(W_2) \equiv l(D'') \equiv 1 \ (mod\ 2). \tag{5}$$

With (4) and (5), we consider two cases.

Case 2.1. Suppose $l(W_2) \equiv 0$ (*mod* 2) and $l(W_1) \equiv l(W_3) \equiv 1$ (*mod* 2). By Lemma 4.3, W_1 contains an odd length circuit X_1. Since the odd-girth of G is at least g,

$$g \leq |E(X_1)| \leq |E(W_1)| \leq l(W_1) = l(P_1) + l(Q_2) + l(Q_1). \tag{6}$$

Note that

$$g = l(D') = l(P_1) + l(P_2) + l(Q_3) + l(Q_1). \tag{7}$$

Combining (6) and (7), we have that

$$l(P_2) + l(Q_3) \leq l(Q_2). \tag{8}$$

Symmetrically, by comparing W_3 and D'', we have that

$$l(P_2) + l(Q_2) \leq l(Q_3). \tag{9}$$

The sum of the above two inequalities ((8) and (9)) yields that $2l(P_2) \leq 0$. This contradicts that $P_2 = v_bCv_c$ is of positive length since $b \neq c$.

Case 2.2. Suppose that $l(W_2) \equiv 1$ (*mod* 2) and $l(W_1) \equiv l(W_3) \equiv 0$ (*mod* 2). We claim that W_0 is an odd circuit of length g. Since

$$l(W_0) = l(W_1) + l(W_3) + l(W_2) - 2l(Q_2) - 2l(Q_3)$$

and both $l(W_1)$ and $l(W_3)$ are even, $l(W_0)$ has the same parity as that of $l(W_2)$, which is odd in this case. Since, for each $i \in \{0, 2\}, l(W_i)$ is odd, by Lemma 4.3, the closed walk W_i contains an odd circuit X_i of length at most $|E(W_i)|$ which is at least g. That is,

$$g \leq l(X_i) \leq l(W_i) \tag{10}$$

for each $i \in \{0, 2\}$.
Since

$$g \leq l(W_2) = l(P_2) + l(Q_2) + l(Q_3) \tag{11}$$

and

$$g = l(D') = l(P_1) + l(P_2) + l(Q_3) + l(Q_1), \tag{12}$$

we have that (combining (11) and (12))

$$l(P_1) + l(Q_1) \leq l(Q_2). \tag{13}$$

Note that

$$g = l(D'') = l(P_2) + l(P_3) + l(Q_4) + l(Q_2)$$

and, by (13),

$$g \geq l(P_2) + l(P_3) + l(Q_4) + (l(P_1) + l(Q_1)) = l(W_0). \tag{14}$$

Combining (10) and (14) for $i = 0$, all equalities hold, and, therefore,

$$l(X_0) = l(W_0) = g$$

and, thus, W_0 is the odd circuit X_0. Hence, we have another critical circuit X_0 with pace $d - a$ around C. This contradicts the choice of $D'' = C_h$, which has the largest pace $d - b = p$ among all critical circuits around C. ∎

Note that Lemma 4.1 is a bit more than what we really need in the proof of Theorem 3.1 (case 1.2 is not needed, if one wants only a direct proof of Theorem 3.1). However, Lemma 4.1 is expected to have further applications in many related problems, which is why we extract it out of the proof of Theorem 3.1.

Note that Lemma 4.1 works only for planar graphs for otherwise, in the proof, two critical circuits outside a facial circuit may not intersect each other.

5. PROOF OF THE MAIN THEOREM

The following lemma is used in the proof of Theorem 3.1.

Lemma 5.1. *Let $P = v_0 \cdots v_s$ be a path. If the end-vertices of the path are pre-colored with $c : \{v_0, v_s\} \mapsto Z_{2k+1}$, then the pre-coloring of $\{v_0, v_s\}$ can be extended to a $\frac{2k+1}{k}$-coloring for the entire path P.*

Proof. When $s > 2k$, the color at v_s can first be (arbitrarily) extended to the vertex set $\{v_{2k}, \ldots, v_s\}$ without worrying about the color at v_0. Then we extend the colors at v_0 and v_{2k} to the remaining vertices. Thus, we only need to consider the case when $s = 2k$.

Since k is a generator of the cyclic group $Z_{2k+1} = \{0, \ldots, 2k\}$, we have that $Z_{2k+1} = kZ_{2k+1}$. So, without loss of generality, we may assume that v_0 and v_{2k} are colored with 0 and tk, respectively, where t is an integer: $0 \le t \le 2k$.

Case 1. *t is an even number*. For $0 \le i \le t$, color v_i with $ik \in Z_{2k+1}$; for $t \le i \le 2k$, color v_i with $tk \in Z_{2k+1}$, if i is even, or with $(t-1)k \in Z_{2k+1}$, if i is odd.

Case 2. *t is an odd number*. For $0 \le i \le 2k+1-t$, color v_i with $-ik \in Z_{2k+1}$; for $2k+1-t \le i \le 2k$, color v_i with $-(2k+1-t)k \in Z_{2k+1}$, if i is odd, or with $-(2k-t)k \in Z_{2k+1}$, if i is even. Note that $-(2k+1-t)k$ and tk are the same element in the group Z_{2k+1}. ∎

Lemma 5.1 is sharp, since it is easy to find a pair colors at v_0 and v_s that cannot be extended to the entire path P, if $s \le 2k-1$.

Proof of Theorem 3.1. Let G be a counterexample to the theorem with the least number of vertices. Any parallel edges whenever they occur are considered as a single edge, since this is a vertex coloring problem.

I. By Lemma 5.1, we may assume that G contains no induced path of length at least $2k$.

II. If G has a facial circuit $C = v_0 \cdots v_{r-1}$ of length $r \neq 10k-3$, then, by Lemma 4.1, there is an integer $i \in \{0, \ldots, r-1\}$ such that the new graph G' obtained from G by identifying v_{i-1} and v_{i+1} is still of odd-girth $10k-3$. Furthermore, a $\frac{2k+1}{k}$-coloring of G' can be extended to the original graph G. This contradicts that G is a counterexample. Therefore, we assume that every facial circuit of G is of the same length, $10k-3$.

III. We may assume that the graph G is 2-connected. Otherwise, $\frac{2k+1}{k}$-colorings for all blocks of G can be combined to form a $\frac{2k+1}{k}$-coloring for the entire graph G.

IV. Define the *underlying graph*, $\bar{G}$, of a graph G to be the graph obtained from G by replacing induced paths with single edges.

Since G is 2-connected, applying Euler's formula, we have that the underlying graph $\bar{G}$ has a facial circuit of length at most 5, see [4]. Let C be a facial circuit of $\bar{G}$ with the shortest length (≤ 5). Let C' be the corresponding circuit of C in G.

Since every facial circuit of G is of length $10k-3$, the facial circuit C', which contains at most five vertices of degree greater than or equal to three, must contain an induced path of length at least $2k$. This contradicts I. ∎

6. FUTURE DIRECTIONS

In light of Grötzsch's Theorem, which states that planar graphs with girth at least four are three colorable ([6], or see [7], [1] and [11]), it may be possible to tighten the result of Theorem 3.1.

Problem 6.1. *For each positive integer k, find a function $h(k)$ such that $h(k) < 10k-3$ and every planar graph with odd-girth at least $h(k)$ is $\frac{2k+1}{k}$-colorable.*

How much further can Theorem 3.1 be improved? Note that $h(1) = 5$ by Grötzsch's Theorem, and $h(2) \geq 9$, since it was found by Albertson and Moore [2] that there is a planar heptangulation (which has odd-girth seven) that is not $\frac{5}{2}$-colorable.

Restricting Problem 6.1 to outerplanar graphs, it is easy to show $h(k) = 2k+1$, using induction on the number of faces. This is because an outerplanar graph either has a cut vertex (in which case the induction proceeds easily) or has a facial circuit C_1 containing at most one chord of the exterior circuit C_0. The result follows by first deleting the edges of $C_0 \cap C_1$ and applying the inductive hypothesis to the remaining graph, and then applying Lemma 5.1 to extend the colors at the endvertices of the chord $C_1 \backslash C_0$ to the deleted path.

Note that Theorem 1.1 is for planar graphs. It is natural to consider a further generalization for nonplanar graphs. Unfortunately, Theorem 1.1 is not true in general, i.e., without the condition of planarity: Youngs [13] constructed a family of 4-chromatic graphs in the projective plane with an arbitrary large odd girth. This example indicates that some topological (or other type of) condition is necessary to extend Theorem 1.1 to nonplanar graphs.

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