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# The 3-flow conjecture, factors modulo $k$, and the 1-2-3-conjecture

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ABSTRACT

Let $k$ be an odd natural number $\geq 5$, and let $G$ be a $(6k-7)$-edge-connected graph of bipartite index at least $k-1$. Then, for each mapping $f : V(G) \to \mathbb{N}$, $G$ has a subgraph $H$ such that each vertex $v$ has $H$-degree $f(v)$ modulo $k$. We apply this to prove that, if $c : V(G) \to \mathbb{Z}_k$ is a proper vertex-coloring of a graph $G$ of chromatic number $k \geq 5$ or $k-1 \geq 6$, then each edge of $G$ can be assigned a weight 1 or 2 such that each weighted vertex-degree of $G$ is congruent to $c$ modulo $k$. Consequently, each nonbipartite $(6k-7)$-edge-connected graph of chromatic number at most $k$ (where $k$ is any odd natural number $\geq 3$) has an edge-weighting with weights $1, 2$ such that neighboring vertices have distinct weighted degrees (even after reducing these weighted degrees modulo $k$). We characterize completely the bipartite graph having an edge-weighting with weights $1, 2$ such that neighboring vertices have distinct weighted degrees. In particular, that problem belongs to P while it is NP-complete for nonbipartite graphs. The characterization also implies that every 3-edge-connected bipartite graph with at

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least 3 vertices has such an edge-labeling, and so does every simple bipartite graph of minimum degree at least 3.


## 1. Introduction

Tutte's flow conjectures are a major source of inspiration in graph theory. The 3-flow conjecture says that every 4-edge-connected graph has an orientation such that each vertex has the same outdegree and indegree modulo 3. The main motivation for the 3-flow conjecture is that its restriction to planar graphs is the dual version of (and hence equivalent to) Grötzsch's Theorem that every planar triangle-free graph has chromatic number at most 3. Jaeger's circular-flow-conjecture extends the 3-flow conjecture to orientations of $(2k-2)$-edge-connected graphs that are balanced modulo an odd number $k$. The weak version of Jaeger's conjecture (where $2k-2$ is replaced by a larger function $\alpha(k)$) was proved by Thomassen [17], and it was proved in [10] that $\alpha(k) \leq 3k-3$.

The 3-flow conjecture also relates to graph decomposition. Indeed Bárat and Thomassen [4] proved that the weak 3-flow-conjecture is equivalent to the statement that a graph of sufficiently large edge-connectivity and size divisible by 3 can be edge-decomposed into claws, and they made the stronger conjecture that, for every tree $T$, there exists a smallest number $\beta_T$ such that every simple graph of edge-connectivity at least $\beta_T$ and size divisible by $|E(T)|$ has an edge-decomposition into copies of $T$. Subsequently, the weak 3-flow conjecture has become a theorem [17,10], and significant progress has been made on the conjecture of Bárat and Thomassen based on the weak circular flow conjecture, see [13].

Thomassen [18] introduced $f$-factors modulo $k$ and proved that a bipartite $(6k-7)$-edge-connected graph has such a factor if the sum of $f$-values on the left-hand side is congruent to the sum of $f$-values on the right-hand side modulo $k$. In the present paper we extend this to non-bipartite graphs of sufficiently large bipartite index, in particular to graphs of large chromatic number. We apply this to the 1-2-3-conjecture using only weights $1, 2$. Specifically, we prove that a non-bipartite $(6k-7)$-edge-connected graph of chromatic number at most $k$ admits a weighting of the edges with weights $1, 2$ such that the weighted degrees reduced modulo $k$ are a proper vertex-coloring of the vertices. If $G$ has chromatic number $k$ or $k-1$, where $k$ is an odd number $\geq 7$, then, for each prescribed proper vertex-coloring $c$ with $k$ or $k-1$ colors, the weighting can be chosen such that the weighted degrees reduced modulo $k$ coincide with $c$. In particular, every non-bipartite $(6k-7)$-edge-connected graph of chromatic number at most $k$ satisfies the 1-2-3-conjecture (even without using the weight 3). We also prove that every bipartite graph has an edge-weighting with weights $1, 2$ such that neighboring vertices have distinct weighted degrees, unless the graph has a very special structure which we characterize completely. In particular, such a weighting exists if the graph is 3-edge-connected or if it is simple and has minimum degree at least 3.

## 2. Graph factors modulo $k$

Tutte's 1-factor theorem, and the more general $f$-factor theorem [19] are a cornerstone in graphs factor theory, and are another major source of inspiration in graph theory, see e.g. the monograph by Lovász and Plummer [9].

Recently, an *$f$-factor modulo $k$* was introduced by the first author in [18] as follows.

**Definition 2.1.** Let $G$ be a graph, let $k$ be an integer ($k \geq 2$), and let $f : V(G) \to \mathbb{Z}_k$. A spanning subgraph $H$ of $G$ is called an *$f$-factor modulo $k$* of $G$ if, for each vertex $v$ in $G$,

$$d_H(v) \equiv f(v) \pmod{k}.$$

An obvious necessary condition for the existence of an $f$-factor modulo $k$ is that $\sum_{v \in V(G)} f(v) \in 2Z_k$. For a bipartite graph $G$ with bipartition $\{V_1, V_2\}$, the following stronger condition is necessary.

$$\sum_{v \in V_1} f(v) - \sum_{v \in V_2} f(v) \equiv 0 \pmod{k}. \tag{1}$$

In [18], the first author obtained the following result using the weak circular flow conjecture (now a theorem) [17,10].

**Theorem 2.2.** (Thomassen [18]) *Let $k \geq 3$ be an odd integer, let $G$ be a bipartite graph with bipartition $\{V_1, V_2\}$, and let $f : V(G) \to \mathbb{Z}_k$ be a mapping satisfying Equation (1). If $G$ is $(3k-3)$-edge-connected, then $G$ has an $f$-factor modulo $k$.*

We shall combine this theorem with the following lemma.

**Lemma 2.3.** (Thomassen [16,18]) *Let $q$ be a positive integer and let $G$ be a $(2q-1)$-edge-connected graph. Then the bipartite subgraph of $G$ induced by any maximum edge-cut is $q$-edge-connected.*

**Definition 2.4.** The *bipartite index* bi(G) of a graph $G$ is the smallest number of edges that must be deleted in order to make the graph bipartite. (In other words, the bipartite index is the number of edges outside of a maximum cut.)

The bipartite index was used in [15] to study linkage properties in highly connected graphs.

**Theorem 2.5.** *Let $k$ be an odd number $> 1$. If $G$ is $(6k-7)$-edge-connected with bipartite index at least $k-1$, and $f : V(G) \to \mathbb{Z}_k$ is an arbitrary mapping, then $G$ has an $f$-factor modulo $k$.*

**Proof of Theorem 2.5.** By Lemma 2.3, the subgraph $H$ of $G$ consisting of the edges of a maximum edge-cut between sets $V_1$, $V_2$ say, is $(3k-3)$-edge-connected. Let $G_i$ be the subgraph of $G$ induced by $V_i$ $(i = 1, 2)$. Since $G$ has bipartite index at least $k-1$, it follows that $|E(G_1)| + |E(G_2| \geq k-1$. Let $M$ be a subgraph of $G$ consisting of some (possibly none) edges in $G_1 \cup G_2$, and no edges in $H$. Now define, for each vertex $v$, $f'(v) = f(v) - d_M(v)$, where $d_M(v)$ is the degree of $v$ in $M$. If $H$ has an $f'$-factor modulo $k$, then that subgraph together with $M$ is an $f$-factor modulo $k$ of $G$. So, it suffices to prove that $M$ can be chosen such that the left-hand side of Equation (1) (with $f'$ instead of $f$) is 0 modulo $k$. If an edge in $G_2$ is added to $M$, then the left-hand side of Equation (1) is increased by 2. The same is true if an edge in $G_1$ is deleted from $M$. So, if we start by letting $M$ consist of all edges in $G_1$ and no edges in $G_2$, and then add edges, one by one to $G_2$, and delete edges, one by one from $G_1$, we see that we can get $k$ distinct left-hand sides in Equation (1). In particular, $M$ can be chosen such that $f'$ satisfies Equation (1) and we are done. This completes the proof of Theorem 2.5.

**Theorem 2.6.** *Let $k$ be an odd number, let $k \geq 5$, and let $G$ be a graph with chromatic number $k$ or (if $k \geq 7$) $k-1$. Let $f : V(G) \to \mathbb{Z}_k$ be an arbitrary mapping. If $G$ is $(6k-7)$-edge-connected, then $G$ has an $f$-factor modulo $k$.*

**Proof of Theorem 2.6.** Let $V_1$, $V_2$, $G_1$, $G_2$ be as in the proof of Theorem 2.5. Then clearly,

$$\chi(G_1) + \chi(G_2) \geq \chi(G) \geq k-1 \geq 4.$$

In a proper vertex-coloring of a graph with the smallest number of colors, there must be an edge between each color class. Hence

$$bi(G) \geq \binom{\chi(G_1)}{2} + \binom{\chi(G_2)}{2} \geq k-1.$$

Thus the statement of Theorem 2.6 follows from Theorem 2.5.

Theorem 2.5 is best possible in the sense that the lower bound $k-1$ on the bipartite index cannot be replaced by $k-2$ even if the condition on the edge-connectivity is sharpened. In fact, for any graph of bipartite index $k-2$ there is a mapping $f$ such that the graph has no $f$-factor modulo $k$. To see this we take a bipartite graph $H$ and add $k-2$ edges. Then there is a mapping $f$ such that the left-hand side in Equation (1) (with $f'$ instead of $f$) is never 0 modulo $k$. In the same way we see that Theorem 2.6 does not hold when $G$ has chromatic number 3 or 4. But is easy to describe completely the exceptions. If $G$ is 3-chromatic, then the proof of Theorem 2.6 applies unless $G$ has an edge $e$ such that $G - e$ is bipartite. Since $G - e$ has large edge-connectivity it is easy to see that $e$ is the only such edge (because $G$ has two odd cycles which have $e$ and no other edge in common). If $G$ is 4-chromatic, then the proof of Theorem 2.6 applies unless $G$ has two (respectively three) edges $e_1$, $e_2$ (respectively $e_1$, $e_2$, $e_3$) such that $G - e_1 - e_2$ (respectively $G - e_1 - e_2 - e_3$) is bipartite. These two (or three) edges are unique. For, if we delete these edges the resulting graph is 4-edge-connected. So if we delete three edges and $e_1$, say, is not one of them, then there still is an odd cycle containing $e_1$.

## 3. Applications to the 1-2-3-conjecture

Let $G$ be a graph. If $v$ is a vertex, then $E(v)$ is the set of edges incident with $v$. The *$G$-degree* $d_G(v)$, or just *degree* $d(v)$ when no confusion is possible, is the cardinality of $E(v)$. If all neighbors of $v$ have degree at most $d(v)$, then $v$ is a *local maximum*. Strict local maximum, local minimum, and strict local minimum are defined analogously, also in the weighted case.

A *neighbor-distinguishing edge-weighting* is an assignment $w : E(G) \to \{1, 2, \ldots\}$ such that the *induced labeling* $w : V(G) \to \mathbb{Z}^+$, where $w(v) = \sum_{e \in E(v)} w(e)$, is a proper vertex-coloring of $G$ (see [7]).

**Conjecture 3.1.** (Karoński, Łuczak and Thomason [7]) *Every connected simple graph of order at least* 3 *has a neighbor-distinguishing edge-weighting with weights* $1, 2, 3$.

This conjecture has become known as the *1-2-3-conjecture* and has attracted considerable attention, see e.g. [1–3,20,6,14].

In this section we apply Theorem 2.6 to verify a special case of Conjecture 3.1 (using only weights $1, 2$), namely for graphs whose edge-connectivity is at least 6 times the chromatic number. In [3] a similar result was proved for simple graphs whose minimum degree is at least 12 times the chromatic number, and in [12] the constant 12 is replaced by 8. Our result has some additional features. First of all, it applies to graphs with multiple edges which is not the case for the above-mentioned results in [3,12], as shown by the odd multi-cacti defined in Section 5. Also, if a graph has chromatic number $k \geq 5$ and large edge-connectivity, then any proper vertex-$k$-coloring can be obtained from an edge-weighting with weights $1, 2$ by reducing the weighted degrees modulo $k$. Here it is not possible to replace large edge-connectivity by large minimum degree, even for simple graphs, as shown by the disjoint union of a large complete graph and an appropriate large complete bipartite graph.

We need the following notation. Let $\mu$ be a positive integer. Let $T_{2,\mu}$ be a multigraph obtained from the complete bipartite graph $K_{2,\mu}$ with bipartition $\{V_1, V_2\}$ that $|V_1| = 2$, $|V_2| = \mu$, and adding a single edge $e_0$ joining both vertices of $V_1$, and replacing each edge of $K_{2,\mu}$ with parallel edges of arbitrary positive multiplicity. Notice that the chromatic number of $T_{2,\mu}$ is 3 and its edge-connectivity can be arbitrarily large.

**Theorem 3.2.** *Let $k$ be an odd number $> 1$, and let $G$ be a nonbipartite $(6k-7)$-edge-connected graph with chromatic number $k$ or $k-1$. Then $G$ has an edge-weighting with weights $1, 2$ such that its induced vertex-labeling reduced modulo $k$ is a proper vertex-coloring unless $G$ is a 3-chromatic graph of the form $T_{2,\mu}$ where $\mu \equiv 2 \pmod 3$. Moreover, if $\chi(G) \geq 5$ and $c : V(G) \to \{1, 2, \dots, k\}$ is a vertex-coloring (not necessarily proper), then the edge-weighting with weights $1, 2$ can be chosen such that its induced vertex-labeling is congruent to $c$ modulo $k$.*

**Proof of Theorem 3.2.** Let $c : V(G) \to \{1, 2, \dots, k\}$ be a vertex-coloring. Define

$$f_c(v) \equiv 2d_G(v) - c(v) \pmod k.$$

If $\chi(G) \geq 5$ and $G$ is $(6k-7)$-edge-connected, it follows from Theorem 2.6 that $G$ has an $f_c$-factor modulo $k$ which we denote by $F$.

Then let

$$w(e) = \begin{cases} 1 & \text{if } e \in E(F); \\ 2 & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} w(u) &= \sum_{e \in E(u)} w(e) = d_G(u) + d_{G-E(F)}(u) \\ &= 2d_G(u) - d_F(u) \equiv 2d_G(u) - f_c(u) \equiv c(u) \pmod k. \end{aligned}$$

This completes the proof when $\chi(G) \geq 5$.

We now consider the case where $G$ has chromatic number 3 or 4. We may also assume that the bipartite index of $G$ is at most $k-2$, by Theorem 2.5. We now refer to the proof of Theorem 2.5. Assume first that $k = 5$ and $G$ has chromatic number 4. We consider a maximum cut with sides $V_1$, $V_2$. As the bipartite index of $G$ is at most 3 we can color $G$ properly such that $V_1$ gets colors $1, 2$ and $V_2$ gets colors $3, 4$ or (if the three edges not in the maximum cut form a triangle in $V_1$, say) $V_1$ gets colors $1, 2, 4$ and $V_2$ gets color 3. Moreover, if $G$ has at least 5 vertices we can choose the coloring such that at least two vertices in $V_2$ have color 3 (unless $V_2$ has only one vertex in which case we recolor such that $V_1$ has at least two vertices of color 3). Each of these colors can be replaced by the color 5. Each of these replacements has the same effect as adding an edge to $G_2$ in the proof of Theorem 2.5. So, if $G$ has chromatic number 4 there only remains the case where $G$ has precisely 4 vertices and hence $G$ is obtained from $K_4$ by adding edges so that the edge connectivity is at least $(6k-7) = 17$. We leave this case for the reader. If $k = 3$ we argue similarly. Recall that $G$ is nonbipartite, by assumption, and hence $G$ has chromatic number 3. Now the maximum cut contains all edges except precisely one which is in $V_1$, say. The proof is now completed as in the previous case if we can color

$G$ in colors $1, 2, 3$ such that $V_2$ (or $V_1$ since we may interchange between $V_1$, $V_2$ in the proof of Theorem 2.5) has at least one vertex of color 1 which is not joined to any vertex of color 3. This is possible unless $V_1$ contains only two vertices joined by an edge and each vertex in $V_2$ is joined to both of the vertices in $V_1$. So, $G$ is a 3-chromatic graph of the form $T_{2,\mu}$. Let $V_1 = \{x_1, x_2\}$ and $e_0 = x_1x_2$. If $\mu \equiv 0 \pmod 3$, then we give $e_0$ weight 1, and we give another edge $yx_1$ the weight 1. We give $yx_2$ weight 0 modulo 3 (which is possible because we may assume that $yx_2$ has multiplicity at least 3). All other edges are given weight 2 modulo 3. If $\mu \equiv 1 \pmod 3$ we do the same except that now $e_0$ gets weight 2. If $\mu \equiv 2 \pmod 3$, the weighting is not possible as Proposition 3.3 below shows. This completes the proof of Theorem 3.2.

We now show that the labeling in the exceptional case in Theorem 3.2 is not possible.

**Proposition 3.3.** *Let $G = T_{2,\mu}$ with $\mu \equiv 2 \pmod 3$. Then $G$ does not admit an edge-weighting with weights $1, 2$ such that its induced vertex-labeling reduced modulo 3 is a proper vertex-coloring. However, $G$ has a neighbor-distinguishing edge-weighting with weights $1, 2$.*

**Proof of Proposition 3.3.** We use the notation in the definition of $T_{2,\mu}$. In particular, let $e_0 = x_1x_2$ where $V_1 = \{x_1, x_2\}$.

Assume (reductio ad absurdum) that $G$ admits a 2-edge-weighting $w : E(G) \mapsto \{1, 2\}$ such that the induced vertex-labeling $f : V(G) \to Z_3$ is a proper vertex-coloring. Then, for any $y \in V_2$, we have that $\{f(x_1), f(x_2), f(y)\} = \{0, 1, 2\}$. Thus, $f(x_1) + f(x_2) + f(y) \equiv 0 \pmod 3$, and, therefore,

$$f(x_1) + f(x_2) \equiv 2f(y) \pmod 3. \tag{2}$$

Since $G - e_0$ is bipartite, $(f(x_1) - w(e_0)) + (f(x_2) - w(e_0)) = \sum_{e \neq e_0} w(e) = \sum_{v \in V_2} f(v)$. That is,

$$f(x_1) + f(x_2) - 2w(e_0) \equiv \sum_{v \in V_2} f(v) \equiv |Y| f(y) \pmod 3. \tag{3}$$

(We here use the fact that $f$ is constant on $V_2$.)

Since $|V_2| \equiv 2 \pmod 3$, by substituting (2) into (3),

$$2f(y) - 2w(e_0) \equiv 2f(y) \equiv 0 \pmod 3.$$

That is, $w(e_0) \equiv 0 \pmod 3$, a contradiction.

We now show that $G$ has a neighbor-distinguishing edge-weighting with weights $1, 2$. We first give $e_0$ weight 1 and all other edges weight 2. This works unless $x_1$, $x_2$ have the same degree. If $\mu \geq 5$, we replace a weight 2 incident with a vertex of smallest degree by the weight 1. If necessary, we also increase the weight of $e_0$ from 1 to 2. If $\mu = 2$ we leave the proof for the reader. This completes the proof of Proposition 3.3.

## 4. Bipartite simple graphs of minimum degree at least 3

We say that a graph has the *1-2-property* if it has a neighbor-distinguishing edge-weighting with weights $1, 2$. The problem of deciding if a graph has the 1-2-property is NP-complete [8]. Lu, Yu, and Zhang [12] proved that every 3-connected, bipartite simple graph has the 1-2-property and raised the question of characterizing those bipartite graphs that do not have the 1-2-property. This problem was also mentioned in [11]. We shall here solve this problem. We divide the proof into two parts. In this section we prove that simple bipartite graphs of minimum degree at least 3 have the 1-2-property. The proof of this is based on an appropriate local extremum and does not involve mathematical induction. Also, this proof applies to the more general case where the weights $1, 2$ are replaced by any two nonnegative integers $a$, $b$ of different parity. In the next section we show how to extend the result of this section to a complete characterization of the bipartite graphs without the 1-2-property. That proof involves mathematical induction and case analysis to deal with vertices of degree 2. This characterization does not extend to the case where the weights $1, 2$ are replaced by e.g. $0, 1$.

We begin with a well-known observation, see e.g. [12].

**Lemma 4.1.** *Let $G$ be a connected graph and let $f : V(G) \to Z_2$ be a mapping satisfying $\sum_{v \in V(G)} f(v) \equiv 0 \pmod 2$. Then $G$ contains an $f$-factor modulo 2.*

**Proof of Lemma 4.1.** Let $A = \{v \in V(G) : \ f(v) \equiv 1 \pmod 2\}$. Since $\sum_{v \in V(G)} f(v) \equiv 0 \pmod 2$ it follows that $|A|$ is even. Let $A = \{v_1, \ldots, v_{2t}\}$. Since $G$ is connected, let $P_j$ be a path of $G$ joining $v_{2j-1}$ and $v_{2j}$ for each $j = 1, \ldots, t$. Then the symmetric difference of all $P_j$'s is an $f$-factor modulo 2.

As observed in [12] this lemma implies the following.

**Corollary 4.2.** *Let $G$ be a connected bipartite graph. Then $G$ has the 1-2-property (such that each edge joins a vertex with odd weighted degree with a vertex with even weighted degree) unless each bipartite class has an odd number of vertices.*

**Proof of Corollary 4.2.** Let $V_1$, $V_2$ be the bipartite classes. If $V_1$, say, is even, then we let $f(v) \equiv 1 \pmod 2$ if $v \in V_1$ and 0 otherwise. We let all edges in the $f$-factor have weight 1 and all other edges have weight 2. This proves the Corollary.

The rest of the paper is devoted to the exceptional situation in Corollary 4.2. We first establish an auxiliary result.

**Lemma 4.3.** *Let $q$ be a natural number $\geq 4$. Let $G$ be a connected graph and let $A$ be a set of at most $q$ independent vertices such that each vertex in $A$ has degree at least $q-1$, or, each vertex of $A$, except possibly one, has degree at least $q$. Assume that no vertex of $A$ is incident with a bridge in $G$. Then, for each vertex $a$ of $A$, there is an edge $e_a$*

*incident with $a$ such that the deletion of all $e_a$, $a \in A$, results in a connected graph unless $|A| = q = 4$, all vertices of $A$ have $G$-degree 3, and $G - A$ has six components each of which is joined to two distinct vertices of $A$.*

**Proof of Lemma 4.3.** The proof is by induction on the number of edges of $G$. If $G$ has only two vertices, then $A$ has only one vertex $a$ and any edge incident with $a$ will do.

Consider first the case where $G$ has a cutvertex $z$. Then $G$ is the union of two connected graphs $G_1$, $G_2$ having precisely $z$ in common. If $z$ is not in $A$, then we apply induction to each of $G_1$, $G_2$. If $z$ is in $A$, then we apply induction to the graph $G_1$ or $G_2$ containing the vertex of $A$ of smallest degree (with $A - z$ instead of $A$) and then we apply induction to the other graph $G_1$, $G_2$ with $z$ playing the role of the exceptional vertex. Assume therefore that $G$ is 2-connected.

Let $d$ be any vertex in $A$ of smallest degree. We let $e_d$ be any edge incident with $d$ and let $G' = G - e_d$.

Consider first the case where $q \geq 5$. If $G'$ has no bridge incident with a vertex in $A \setminus \{d\}$, we complete the proof by induction to $G'$ with $A \setminus \{d\}$ instead of $A$. So assume that $G'$ has at least one bridge incident with a vertex in $A \setminus \{d\}$.

We delete from $G'$ all its bridges and try to apply induction to each component of the resulting graph $G''$. As $G$ has no cutvertex, no vertex in $A$ is incident with two bridges in $G'$. Therefore it is indeed possible to use induction when $q \geq 5$.

We are left with the case where $q = 4$. We complete the proof by induction to $G''$ with $A \setminus \{d\}$ instead of $A$, unless some vertex in $A \setminus \{d\}$ has $G''$-degree 2 and hence $G$-degree 3. Then also $d$ has $G$-degree precisely 3, by the minimality property of $d$. Now focus on the graph $G - d$. Since $G$ is 2-connected it follows that $G - d$ is connected, and that $d$ is joined to every maximal bridgeless subgraph of $G - v$ incident with only one bridge. In particular, $G - d$ has at most three maximal bridgeless subgraphs each of which is incident with only one bridge. As the deletion of any edge incident with $d$ leaves a graph with a bridge, it follows that $G - d$ has precisely three maximal bridgeless subgraphs, each incident with only one bridge, and $d$ is joined to each of them. It also follows that $G - d$ minus its bridges has precisely one component, say $M$, incident with precisely three bridges $e_1$, $e_2$, $e_3$, say, in $G - d$. We claim that each of these three bridges in $G - d$ is incident with a vertex of $A$ which has $G$-degree 3. To prove this claim, let us assume that $e_1$ is not incident with a vertex in $A$ which has $G$-degree 3. Now $G$ has a path $P$ from $d$ to an end of $e_1$ not containing any vertex of $M$. Let $e'_1$ be the first edge of this path $P$. As $G - e'_1$ has a bridge incident with a vertex $a$ in $A$ of $G$-degree 3, it follows that $G - e'_1$ minus its bridges has a component $M'$ containing $a$ such that $M'$ is incident with precisely two bridges in $G - d$. If we let $e_a$ be an edge incident with $a$ and not contained in $P$, then $G - e_a$ has no bridge incident with a vertex in $A$. This contradiction proves the claim since otherwise, we complete the proof using induction to $G - e_a$ with $A \setminus \{a\}$ instead of $A$. Hence $A$ has precisely four vertices, each of these has $G$-degree 3, and each vertex in $A$ can play the role of $d$. We have proved that $G - A$

has at least three components, and three of these components are joined to $d$ and some other vertex of $A$. As this holds for any vertex of $A$, the result follows.

**Corollary 4.4.** *Let $G$ be a bipartite simple 3-connected graph with an odd number of vertices in each of the bipartite classes $X$, $Y$. Assume that $x$ is a local maximum in $X$, say, of $G$-degree at least 4. Let $f : V(G) \to Z_2$ be the following mapping: $f(x) = 0$, and $f(y) = 0$ for each vertex $y$ in $Y$, and let $f$ be 1 otherwise. Then $G$ has a spanning subgraph $H$ which is an $f$-factor modulo 2 such that $d_H(x) = 0$ and $d_H(y) > 0$ for each $G$-neighbor $y$ of $x$ for which $d_G(y) = d_G(x)$.*

The proof of Corollary 4.4 follows by first deleting $x$, and then applying Lemma 4.3 to the graph $G - x$ where $A$ is the set of those $G$-neighbors $y$ of $x$ for which $d_G(y) = d_G(x)$. Then we apply Lemma 4.1 to the resulting graph with the new $f$ being defined appropriately.

Note that we obtain a neighbor-distinguishing edge-weighting of $G$ by giving each edge in $H$ the weight 1 and each edge outside $H$ the weight 2. It is this idea we shall refine to characterize the bipartite graphs without the 1-2-property.

**Theorem 4.5.** *Let $G$ be a simple bipartite graph of minimum degree at least 3. Then $G$ has the 1-2-property.*

**Proof of Theorem 4.5.** It suffices to consider the case where $G$ is connected. We may assume that each bipartite class has a odd number of vertices since otherwise, the statement follows from Corollary 4.2. We let $G'$ be an endblock of $G$. Possibly $G' = G$. If $G' \neq G$, we let $x_0$ be the unique cutvertex of $G$ contained in $G'$. If the deletion of any two neighboring vertices in $G'$ of the same $G$-degree leaves a connected graph we put $G'' = G'$. Otherwise, we select an edge $y_0z_0$ in $G'$ such that $y_0$, $z_0$ have the same $G$-degree, $G' - y_0 - z_0$ is disconnected and such that a component $H$ of $G' - y_0 - z_0$ not containing $x_0$ is smallest possible. The union of that component $H$ and $y_0$, $z_0$ and all edges connecting them is called $G''$. Possibly $x_0$ is one of $y_0$, $z_0$. We now choose a local maximum or minimum $w_0$ as follows. If all $G$-degrees of $G''$ are the same, then we let $w_0$ be any vertex which is not a neighbor of any of $x_0$, $y_0$, $z_0$. It is easy to see that such a vertex $w_0$ exists because $G$ is connected. If all $G$-degrees of $G''$ are not the same, then some vertex of $H$ has $G$-degree greater than that of $y_0$, $z_0$ (or $x_0$ if $y_0$, $z_0$ do not exist) or smaller than that of $y_0$, $z_0$ (or $x_0$ if $y_0$, $z_0$ do not exist). We let $w_0$ be such a vertex of maximum (respectively minimum) $G$-degree. Then $w_0$ is a local maximum or a local minimum. We give all edges incident with $w_0$ the weight 2 if $w_0$ is a local maximum and all edges incident with $w_0$ get weight 1 if $w_0$ is a local minimum. We shall extend this weighting to $G$ such that $w_0$ and all its neighbors have the same weighted degree modulo 2, and all vertices $\neq w_0$ in the same bipartite class as $w_0$ have a weighted degree opposite to that of $w_0$, modulo 2, and all vertices in the other bipartite class have the same weighted degree as $w_0$ modulo 2. This will prove the theorem except that possibly $w_0$ and one of its neighbors may have the same weighted degree. We shall avoid this by using Lemma 4.3. We shall make $w_0$ a strict local maximum (or minimum) in the weighted sense. This is the same as saying that a neighbor $a$ of $w_0$ which has the same $G$-degree as $w_0$ does not have all incident edges of the same weight. Note that $a$ is distinct from $x_0$, $y_0$, $z_0$ by the choice of $w_0$. Also, $G - w_0 - a$ is connected by the minimality of $G''$. In other words, $a$ is not a cutvertex in $G - w_0$.

Consider first the case where $w_0$ has $G$-degree at least 4. By Lemma 4.1, $G - w_0$ has a spanning subgraph $H$ such that when we give all its edges weight 1 and all edges in $G - w_0$ outside $H$ weight 2 and then add $w_0$ and all its weighted edges to $H$, then we obtained a weighted graph such that $w_0$ and all its neighbors have the same weighted degree modulo 2 and, for any other edge, the ends have different weighted degrees modulo 2. Now the only problem is that some neighbor $a$ of $w_0$ has all incident edges of the same weight. We apply Lemma 4.3 to avoid this. More precisely, before we use Lemma 4.1 we delete an edge $e_a$ for each neighbor of $w_0$ with the same $G$-degree as $w_0$. If all edges incident with $w_0$ have weight 1 we make sure that $e_a$ is outside $H$ so that $e_a$ gets weight 2. If all edges incident with $w_0$ have weight 2 we make sure that $e_a$ is inside $H$. We achieve this by changing parity of all ends of edges of the type $e_a$ before we apply Lemma 4.1. If $w_0$ has $G$-degree precisely 4, then we try to ensure that the exceptional situation in Lemma 4.3 does not occur. We can achieve this if we can choose $w_0$ such that some neighbor of $w_0$ has $G$-degree distinct from that of $w_0$. This is clearly possible unless all vertices in $G''$ have $G$-degree precisely 4. We may also assume that, if we delete $w_0$ and all its neighbors, then the resulting graph has six components. However, if we choose $w_0$ such that the deletion of $w_0$ and all its neighbors has a component (containing one of $x_0$, $y_0$, $z_0$ if they exist) which is maximum, then the other components are easily seen to be isolated vertices and there can be at most three such components. This completes the proof if $w_0$ has degree at least 4.

We are left with the case where $w_0$ must have $G$-degree 3. We let $M$ be the graph obtained from $G$ by deleting $w_0$ and all those neighbors of $G$ that have $G$-degree 3. If $M$ is connected, then we can use Lemma 4.1 to find a spanning subgraph with $w_0$ and its three neighbors having odd degree, and for any other edge, the ends have distinct degrees modulo 2. Moreover, $w_0$ has degree 3 in this subgraph, and each neighbor of $w_0$ of $G$-degree 3 has degree precisely 1 in the subgraph and hence weighted degree 5. So we are left with the case where $M$ must be disconnected. We chose $w_0$ such that the component of $M$ containing one of $y_0$, $z_0$, $x_0$ is maximum. (If $y_0$, $z_0$, $x_0$ do not exist, we just maximize some component.) If some other component of $M$ has a vertex of $G$-degree at least 4, then we choose one, say $w_0'$ of maximum degree, and we repeat the argument where $w_0$ has degree at least 4. If it is not possible to chose such a $w_0'$, then all vertices in the components of $M$ not containing any of $x_0$, $y_0$, $z_0$ have $G$-degree 3. The maximality property of $M$ implies that all components of $M$, except one, are an isolated vertex. There can be only one such isolated vertex (unless $G = K_{3,3}$). So, all neighbors of $w_0$ have $G$-degree 3, and there is a vertex $w_0' \neq w_0$ having the same three neighbors as $w_0$.

But now it is easy to obtain the desired spanning subgraph of $G$ by applying Lemma 4.1 to the connected graph $M - w'_0$.

This completes the proof of Theorem 4.5.

## 5. Bipartite graphs without the 1-2-property

In [5] there are examples of bipartite graphs without the 1-2-property. In this section we give a complete characterization. We shall present the exceptional graphs in two ways. Clearly, a graph with two vertices does not have the 1-2-property. Let $q$ be an odd natural number $> 1$, and let $G_1, G_2, \ldots, G_q$ be pairwise disjoint graphs each with an edge $x_i y_i$ of multiplicity at least 2. Delete one of the edges between $x_i y_i$ for $i = 1, 2, \ldots, q$. Add the edges $y_i x_{i+1}$ for $i = 1, 2, \ldots, q-1$. Add also the edge $y_q x_1$. We say that the resulting graph $G$ is an *odd combination* of $G_1, G_2, \ldots, G_q$. If $G$ has the 1-2-property, then clearly at least one of $G_1, G_2, \ldots, G_q$ has the 1-2-property because some two consecutive edges in the cyclic sequence $y_1 x_2, y_2 x_3, \ldots, y_q x_1$ must have the same weight. So if we start with the graphs with two vertices joined by a multiple edge and take odd combinations, then we obtain a class of bipartite graphs without the 1-2-property. Let us call such a graph an *odd multi-cactus*. (Here "odd" refers to the number of green edges in each building stone replaced by a multiple edge in the description below.)

These graphs can also be described in another way as follows. Take a collection of simple cycles each of length 2 modulo 4 and each with edges colored alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity $\geq 1$. The graph with one edge and two vertices is also called an odd multi-cactus.

We briefly indicate why the two descriptions are equivalent. When we form $G$ as an odd combination of $G_1, G_2, \ldots, G_q$, then we may assume, by induction, that each $G_i$ is obtained by pasting cycles together along edges. We may assume that we start with the cycle containing the edge $x_i y_i$. But then $G$ is obtained in the same way starting with the cycle of length $2q$ (and every second edge doubled). By a similar argument the graphs obtained by pasting cycles together can be thought of as graphs obtained by taking odd combinations.

In a simple cycle of length 2 modulo 4 we can interchange the colors red and green. However, once we have pasted two cycles together or replaced an edge by a multiple edge, the red-green coloring is unique. If the red-green coloring is unique and we replace a red edge by an edge of multiplicity $> 1$ we obtain a graph with the 1-2-property. We leave the proof of this observation for the reader.

**Theorem 5.1.** *Let $G$ be a connected bipartite graph without the 1-2-property. Then $G$ is an odd multi-cactus.*

**Proof of Theorem 5.1.** The proof is by induction on the number of edges of $G$. Suppose (reductio ad absurdum) that Theorem 5.1 is false and let $G$ be a counterexample with as few vertices as possible and (subject to this) as few edges as possible. Clearly, $G$ has at least three vertices. Let $X$, $Y$ be the bipartition of $G$. Corollary 4.2 implies that each of $X$, $Y$ has odd cardinality. We now establish a number of properties of $G$ which will eventually lead to a contradiction.

(1). Each vertex of $G$ has at least two neighbors.

**Proof of (1).** Suppose (reductio ad absurdum) that $x \in X$ has only one neighbor $y$. By Corollary 4.2, $G - x$ has an edge-weighting such that each vertex in $X - x$ has odd weighted degree and each vertex in $Y$ has even weighted degree. Now let all edges between $x$ and $y$ have weight 2. Then any two neighboring vertices have weighted degrees of distinct parity except $x$, $y$. But, $y$ has larger weighted degree than $x$. This contradiction proves (1).

It follows that each of $X, Y$ has odd cardinality $\geq 3$. It also follows that each endblock of $G$ has at least 4 vertices.

(2) $G$ has at least one vertex having at least 3 neighbors.

**Proof of (2).** If all vertices of $G$ have precisely two neighbors, then $G$ is obtained from a cycle by replacing some edges by multiple edges. If this cycle has length 2 modulo 4 and every second edge has multiplicity 1, then clearly $G$ is an odd multi-cactus. If $G$ does not have this structure, then some two edges of distinct colors red/green are replaced by edges of multiplicity $> 1$. But then $G$ has the 1-2-property. We leave the proof of this for the reader.

We define a *$G$-suspended path or cycle* as a path or cycle $x_1x_2 \dots x_q$ such that all intermediate vertices have $G$-degree 2, and $x_1$, $x_q$ have $G$-degree at least 3. All vertices $x_1, x_2, \dots, x_q$ are distinct, except that possibly $x_1 = x_q$.

(3) An endblock of $G$ cannot contain a suspended path of length 2.

**Proof of (3).** Suppose (reductio ad absurdum) that $y_1xy_2$ (where $x \in X$) is a suspended path in an endblock of $G$. As $x$ is in an endblock, $G - x$ is connected. By Lemma 4.1, $G - x$ has an edge-weighting such that $y_1$, $y_2$ and all vertices of $X \setminus \{x\}$ have odd weight and all vertices $Y \setminus \{y_1, y_2\}$ have even weight. Then add $x$ and give the edges incident with $x$ the weight 1. Then every edge joins two vertices whose weights have distinct parity, except the edges $xy_1$, $xy_2$. However, the weighted $G$-degrees of $y_1$, $y_2$ are strictly greater than the weighted $G$-degree of $x$.

(4) An endblock of $G$ cannot contain a suspended path or cycle of length 4.

**Proof of (4).** Suppose (reductio ad absurdum) that $y_1x_1y_2x_2y_3$ (where $x_1 \in X$) is a suspended path (or cycle) in an endblock of $G$. If possible, we choose this path such that $y_3$ is not a cutvertex in $G$. We delete $x_1$, $x_2$, $y_2$, $y_3$. If $y_3$ is not a cutvertex in $G$, then $G - x_1 - x_2 - y_2 - y_3$ is connected and has therefore, by Lemma 4.1, an edge-weighting such that $y_1$ and all vertices of $X \setminus \{x_1, x_2\}$ have odd weighted degree, and all other vertices have even weighted degree. Then we give the edges of the path $y_1x_1y_2x_2y_3$ the weights $1, 1, 2, 2$, and we give all edges incident with $y_3$ the weight 2. If $y_3$ is a non-cutvertex, then this results in a neighbor-distinguishing edge-weighting. So we may assume that $y_3$ is a cutvertex, and similarly, $y_1$ is a cutvertex. In particular, $y_1 = y_3$. We now repeat the argument except that we delete $x_1$, $x_2$, $y_2$ before we use Lemma 4.1. This results in an neighbor-distinguishing edge-weighting unless $y_1$, $x_2$ get weighted degree 4. This implies that $y_1$ has $G$-degree 3. Let $x_3$ be the third neighbor of $y_1$. Now we apply Lemma 4.1 to give an edge-weighting of $G - x_1 - x_2 - y_1 - y_2$ such that $x_3$ and all vertices of $Y$ have odd weighted degree and all other vertices have even weighted degree. We give $x_3y_1$ weight 1 and we give the edges in $y_1x_1y_2x_2y_3$ weights $2, 1, 1, 2$. As $x_3$ now has even weighted degree, this edge-weighting is neighbor-distinguishing, a contradiction which proves (4).

(5) An endblock of $G$ cannot contain a suspended path or cycle of length at least 5.

**Proof of (5).** Suppose (reductio ad absurdum) that $y_1x_1y_2x_2y_3x_3$ (where $x_1 \in X$) is a path in an endblock of $G$ such that the $G$-degree of each of $x_1$, $y_2$, $x_2$, $y_3$ is 2. Now delete each of $x_1$, $y_2$, $x_2$, $y_3$ and add an edge $y_1x_3$ even if there is already such an edge. If the resulting graph is an odd multi-cactus, then so is $G$. (Note that $y_1x_3$ may be red. But then it has multiplicity 1 in the odd multi-cactus, and the length of the chordless cycle to which it belongs is enlarged by 4 when we put the suspended path back.) Otherwise, the resulting graph has a neighbor-distinguishing edge-weighting with weights $1, 2$. We give $y_1x_1$, $y_3x_3$ the same weight as $y_1x_3$ and delete that edge. We give $y_2x_2$ the opposite color. Then we give $x_1y_2$, $x_2y_3$ distinct colors. There are two choices for this, and one of them will give an neighbor-distinguishing edge-weighting of $G$, a contradiction which proves (5).

It follows from (3), (4), (5) that each suspended path in an end-block has length 3.

(6) An endblock of $G$ cannot contain two vertices $x$, $y$ joined by precisely $q \geq 2$ edges such that each of $x$, $y$ has $G$-degree $q + 1$.

**Proof of (6).** Suppose (reductio ad absurdum) that $x$, $y$ are joined by precisely $q \geq 2$ edges and that $x$ has neighbors $y$, $x_1$ and $y$ has neighbors $x$, $y_1$. Replace the $q$ edges between $x$, $y$ by a single edge. If the resulting graph $G'$ has the 1-2-property, then so does $G$. (The edges $x_1x$, $yy_1$ have the weights $1, 2$, respectively. There are at least 3 possibilities for the weights of the $xy$ edges, and at least one of them gives the desired edge-weighting.) If $G'$ does not have the 1-2-property, then it is an odd multi-cactus with green and red edges. The only problem is that $xy$ may be a red edge. But, as we noted after the definition of an odd multi-cactus, it is easy to prove that, if you replace a red edge in an odd multi-cactus by a multiple edge, then the resulting graph has the 1-2-property (unless the odd multi-cactus is a simple cycle in which you can interchange between red and green colors).

We now repeat the proof of Theorem 4.5. As in the proof of that theorem we define $G'$, $x_0$, $y_0$, $z_0$, $H$, $G''$. Before we define $w_0$ we modify the graph $G$ as follows. Note that $G''$ is contained in an endblock of $G$. Therefore, every suspended path in $G''$ has length 3, by (3), (4), (5). We replace each such path by a new edge which we call a *blue edge*. All other edges of $G$ are *white edges*. We call the resulting graph the *white-blue-graph*. In this graph we now define $w_0$ as in the proof of Theorem 4.5 with the only exception that $w_0$ may be a neighbor of one of $x_0$, $y_0$, $z_0$ because we now allow multiple edges. As in the proof of Theorem 4.5 we give all edges incident with $w_0$ the weight 2 if $w_0$ is a local maximum and all edges incident with $w_0$ get weight 1 if $w_0$ is a local minimum. We shall extend this weighting to $G$. Any blue edge corresponds to a path with 3 edges which will receive the colors $1, 1, 2$ or $1, 2, 2$ or $2, 1, 1$ or $2, 2, 1$, respectively, so that the intermediate vertices of $G$-degree 2 get weights $2, 3$ or $3, 4$ or $3, 2$ or $4, 3$, respectively. This flexibility is convenient when there are blue edges. In the proof of Theorem 4.5 it is important that the deletion of $w_0$ does not create a cutvertex which is a neighbor of $w_0$ and which has the same $G$-degree as $w_0$. This may happen in the present proof when the cutvertex is joined to $w_0$ by a blue edge. But we need not worry about this because of the flexibility of the blue edge. (We explain this in more detail at the end of the proof.) It is only the vertices joined to $w_0$ by white edges that represent a problem.

Let $w_0 \in X$. As in the proof of Theorem 4.5 we shall give $G$ an edge-weighting such that the vertices of $Y$ of $G$-degree at least 3 all have the same parity of degree, and $w_0$ is the only vertex in $X$ of $G$-degree at least 3 whose degree has that parity. The vertices of degree 2 correspond to suspended paths of length 3, and the two intermediate vertices on each such path get weighted degrees $2, 3$, or $3, 4$, respectively. And we can interchange these weights by changing the weight of the edge between them. (This is the afore-mentioned flexibility of the blue edges.)

We now discuss how to modify the proof of Theorem 4.5. We consider the case where $w_0$ is a local minimum in the blue-white graph. (The case where $w_0$ is a local maximum is similar.) If all edges incident with $w_0$ are white and of multiplicity 1, the proof is as the proof of Theorem 4.5. So we may assume that either $w_0$ is incident with at least one blue edge or at least one edge of multiplicity at least 2 or both. If there are multiple edges incident with $w_0$, then $w_0$ has fewer neighbors than the degree, and therefore it is even easier to apply Lemma 4.3. If $w_0$ is incident with a blue edge which in $G$ corresponds to the path $w_0cba$, then the edges of that path will get the weights $1, 1, 2$, and therefore the ends of each edge in this path will get distinct weights because $a$ gets a weight strictly greater than that of $w_0$. If $w_0$ is joined to one of $y_0z_0$, say $y_0$, by a white edge, then we ensure that $w_0$ and $y_0$ get distinct weights by making sure that $y_0z_0$ gets weight 2. (This is possible because $G' - y_0z_0$ is an endblock of $G - y_0z_0$.)

These arguments complete the proof except in the case where $w_0$ has degree 3 in $G$ and also in the white-blue graph. We shall consider the most difficult cases. The first case is when $w_0$ is joined to $a$, say, with a white edge of multiplicity 1 and to $b$, say, with two white edges. But then (6) implies that the weighted degree of $b$ will automatically be $> 3$, and therefore we only need to ensure that $a$ gets weighted degree $> 3$ which is done as in the proof of Theorem 4.5 because $a$ is not a cutvertex in $G - w_0$, by the minimality of $H$.

Our second case is when $w_0$ is a local minimum of degree 3 joined to distinct vertices $a$, $b$, $c$ such that the edges $w_0b$, $w_0c$ are white, and $w_0a$ is blue. Note that possibly $G - w_0 - a$ is disconnected. The blue edge corresponds to a path $w_0yxa$. We shall find a spanning subgraph $F$ such that all vertices in $(Y \setminus \{y\}) \cup \{x, w_0\}$ have odd degree and all other vertices have even degree. Moreover, $F$ should contain all edges incident with $w_0$ and also the edge $yx$ but not the edge $xa$. If $b$ has $G$-degree 3, $F$ should contain the edge $bw_0$ and none of the other two edges. Similarly for $c$. Assume that each of $b$, $c$ has $G$-degree 3, the most difficult case. Then we give all edges in $F$ the weight 1 and all other edges weight 2. All edges except $w_0b$, $w_0c$, $xa$ join vertices whose weights are distinct modulo 2. Moreover, the choice of $H$ implies that $w_0$, $x$ have weight 3 whereas $a$, $b$, $c$ have weight $> 3$. Clearly $F$ exists if $G - w_0 - b - c$ is connected. On the other hand, if $G - w_0 - b - c$ is disconnected, then we focus on the smallest component not containing $x_0$, $y_0$, $z_0$ and we repeat the argument from Theorem 4.5.

This completes the proof of Theorem 5.1.

## 6. Concluding remarks

We have provided a good characterization of the bipartite graphs without the 1-2-property. As mentioned earlier, Khatirinejad et al. [8] proved that this problem is NP-complete for nonbipartite graphs. However, the problem may be polynomial even for nonbipartite graphs if we add some mild conditions on the connectivity or edge-connectivity. In Theorem 3.2 the edge-connectivity needed for the 1-2-property is a linear function of the chromatic number. The complete simple graphs show that a linear function is indeed needed.

It follows from Theorem 5.1 that every connected bipartite graph with a cutvertex has the 1-2-property. The following example shows this is false for nonbipartite graphs, even if the edge-connectivity is large. The example is based on the following observation: If $G$ is a simple graph with $n$ vertices, where $n \geq 3$ is odd, and $G$ has only one pair of vertices of the same degree, then that degree is $(n-1)/2$. This is easy to prove by deleting two vertices, one of smallest degree and one of largest degree, and then use induction. Now take the disjoint union of two large complete graphs $K_n$ where $n$ is odd. Select a vertex $x$ in one of them and a vertex $y$ in the other. Then add $10^{10}$ edges between $x$ and $y$. The resulting graph $G$ does not have the 1-2-property. For suppose it has an neighbor-distinguishing edge-weighting with weights $1, 2$. Let $H$ be the subgraph of the $K_n$ containing $x$ and consisting of all edges of weight 1. Then $H$ has degrees $1, 2, \ldots, n-1$ or $0, 1, \ldots, n-2$, and precisely two vertices of $H$ have the same degree $(n-1)/2$. One of these vertices is $x$. Then the weighted $G$-degree of $x$ is between $3(n-1)/2 + 10^{10}$ and $3(n-1)/2 + 2 \cdot 10^{10}$, but all those weighted degrees are also present in $K_n - x$, a contradiction.

We conclude by pointing out that many of our results on the 1-2-property extend to the a-b-property, defined in the obvious way. Theorem 3.2 says, among other things, the following: Let $k$ be an odd number $\geq 5$, and let $G$ be a nonbipartite $(6k-7)$-edge-connected graph with chromatic number $k$ or (if $k \geq 7$) $k-1$. Let $c : V(G) \to \{1, 2, \ldots, k\}$ be any vertex-coloring (not necessarily proper). Then $G$ has an edge-weighting with weights $1, 2$ such that its induced vertex-labeling is congruent to $c$ modulo $k$. If $F$, $H$ denote the edges of weights $1, 2$, respectively, then, for each vertex $v$,

$$d_F(v) + 2d_H(v) \equiv c(v) \pmod{k}.$$

We let $k$, $G$ be as above, but now we consider two integers $a$, $b$ such that $0 \leq a < b$, and $gcd(b-a, k) = 1$. Now let $c$ be any vertex-coloring (not necessarily proper) of $G$. We seek an edge-decomposition of $G$ into two graphs $F$, $H$ such that, for each vertex $v$,

$$ad_F(v) + bd_H(v) \equiv c(v) \pmod{k}$$

or, equivalently,

$$(a-b)d_F(v) + bd_G(v) \equiv c(v) \pmod{k}.$$

As $gcd(b-a, k) = 1$ we can solve this equation with respect to $d_F(v)$ and denote the solution $f(v)$. We now apply Theorem 2.6 to find an $f$-factor modulo $k$.

Theorem 4.5 extends to the a-b-property when $a$, $b$ are any nonnegative integers of distinct parity. In particular, it extends to the 0-1-property which, together with Theorem 4.5, settles Problem 2 in [11].

Lu [11] gave the following examples of graphs with the 1-2-property but not the 0-1-property: Take the disjoint union of two 6-cycles (or, more generally, we may take any two odd multi-cacti) and join them by a path of length 3. More generally, we can use a path of length 3 modulo 4, and we can iterate the operation. So, Theorem 5.1 does not extend to the 0-1-property. However, Theorem 5.1 extends to the a-b-property whenever $a$, $b$ are natural numbers such that $a < b$, $a$ is odd, and $b$ is even.

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