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Cycle covers (II) – Circuit chain, Petersen chain and Hamilton weights

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ABSTRACT

The Circuit Double Cover Conjecture is one of the most challenging open problems in graph theory. The main result of the paper is related to the characterization of circuit chain structure, which has been one of the most popular approaches to the conjecture. Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \to \{1, 2\}$ such that (G, w) does not have a faithful circuit cover. If, for every weight 2 edge e_0 of (G, w), the eulerian weighted graph $(G - e_0, w)$ has a faithful circuit cover and (G, w) has no removable circuit avoiding e_0, then it was proved (Alspach et al., 1993 [1] or 1994 [2]) that G contains a Petersen minor. It was further conjectured by Fleischner and Jackson (1988) that this graph G must be the Petersen graph. This conjecture was verified *(JCTB 2010)* recently under the assumption of the Hamilton weight conjecture. These two earlier results are further strengthened in this paper as follows. If, for a given weight 2 edge e_0, the eulerian weighted graph $(G - e_0, w)$ has a faithful circuit cover and (G, w) has no removable circuit avoiding e_0, then, under the assumption of the Hamilton weight conjecture, G must be a Petersen chain. With a much weaker requirement *"for a given e_0"* instead of *"for every e_0"*, this strengthened result (structure of circuit chain joining a

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missing edge) is expected to be much more useful in future studies about circuit covering problems.

1. Introduction

All graphs considered in this paper are finite, and may have parallel edges or loops. The following conjecture is one of the most challenging open problems in graph theory.

The Circuit Double Cover Conjecture. *(See Tutte [19], Szekeres [18], Itai and Rodeh [11], Seymour [16], or see [4].) Every bridgeless graph G has a family $\mathcal{F}$ of circuits such that every edge of G is contained in precisely two members of $\mathcal{F}$.*

Since a minimum counterexample to the circuit double cover conjecture is cubic and 3-connected [13], we will discuss circuit covering problems for cubic graphs in most of this paper.

Let G be a smallest counterexample to the circuit double cover conjecture and let $e_0 = x_0y_0 \in E(G)$. Then $G - e_0$ has a circuit double cover $\mathcal{C}$. Let $\mathcal{P}$ ($\mathcal{P} \subseteq \mathcal{C}$) be a circuit chain joining the endvertices x_0, y_0 of the uncovered edge e_0. (A *circuit chain* joining x_0, y_0 is a family of circuits $C_1, \cdots, C_t$ with $x_0 \in V(C_1), y_0 \in V(C_t)$ and $V(C_i) \cap V(C_j) \neq \emptyset$ if and only if $i = j \pm 1$. See Fig. 1.)

> *Can we find a family $\mathcal{Q}$ of circuits of the induced subgraph $G[\{e_0\} \cup \bigcup_{C\in\mathcal{P}} E(C)]$ such that each edge $e \in \bigcup_{C\in\mathcal{P}} E(C)$ is covered by μ members of $\mathcal{Q}$ if e is covered by μ members of $\mathcal{P}$, and the edge e_0 is covered twice?* (For an example see Fig. 2.)

If yes, then $\mathcal{Q} + (\mathcal{C} - \mathcal{P})$ is a circuit double cover of G. This contradicts that G is a counterexample to the CDC conjecture. So, the answer must be "no".

Question 1.1. *What is the structure of the induced subgraph $G[\{e_0\} \cup \bigcup_{C\in\mathcal{P}} E(C)]$?*

This is one of the most popular approaches to the CDC conjecture, originally appearing in [16]. Motivated and promoted by this approach, some related structural studies, new concepts, and techniques have been introduced and studied ([16,1,2], etc.).

Our goal is to determine the structure of the graph described in Question 1.1. The main result (Theorem 4.7) further generalizes some earlier results in [1,2,23], and others. Its relation with the result about minimal contra pairs in [23] will be discussed in detail in Section 4 after the main theorem.

The $(1,2)$-eulerian weighted graph (P_{10}, w_{10}) illustrated in Fig. 4 is the minimum contra pair. It was proved in [2] that every contra pair must contain a Petersen minor.

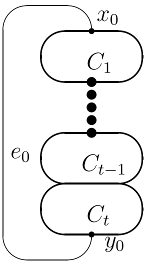

Fig. 1. Circuit chain $\mathcal{P}$ joining the endvertices of e_0.

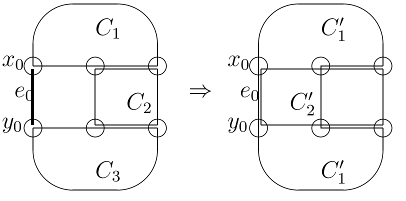

Fig. 2. Circuit cover adjustment for a circuit chain $\{C_1, C_2, C_3\}$ and a missing edge e_0.

However, those pieces of structural information are not enough for further study of or a final attack on the Circuit Double Cover Conjecture. Many conjectures have been proposed for further characterizations of "critical" or "minimal" contra pairs [7,9,10,12,8]. Verification of any of those conjectures would provide much better and clearer structural information for a smallest counterexample to the CDC conjecture.

There are articles/results [6,17,15] providing powerful approaches to find a Petersen minor, but almost no result yet for the determination of the precise structure of a Petersen graph, although many long standing open problems [7,9,10,12,8] demand the precise structure of the Petersen graph (instead of graphs with a Petersen minor, the Petersen graph is expected to be the only exception of those conjectures). The determinations of the Petersen graph structure and the existence of a Petersen minor are significantly different in nature. One of the most important parts of the main theorems in this series of articles is to show that we have the structure of the Petersen graph.

2. Notation and terminology

For notations not defined here see [20], [3], [5] or [22].

Let A and B be two sets. The *symmetric difference* of A and B, denoted by $A \triangle B$, is defined as follows:

$$A \triangle B = (A \cup B) - (A \cap B).$$

Most graphs considered in main theorems, conjectures and lemmas of this paper are cubic. Some subgraphs appearing in the proofs of some theorems or lemmas may have smaller degrees, but their maximum degrees are at most 3.

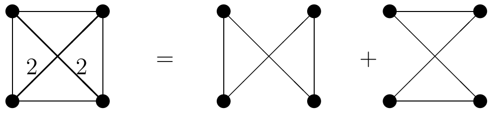

Fig. 3. An example of a faithful cover.

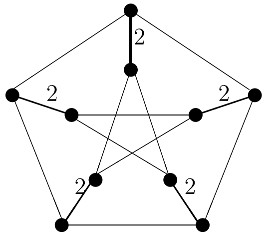

Fig. 4. (P_{10}, w_{10}).

A *circuit* is a connected 2-regular graph, while an *even subgraph* (or cycle) is a graph with even degree for every vertex. An edge e is a *bridge* of a graph G if the removal of e increases the number of components.

Let $G = (V, E)$ be a graph. The *suppressed graph*, denote by $\overline{G}$, is the graph obtained from G by suppressing all degree 2 vertices.

An edge-cut T of G is *trivial* if some component of $G - T$ is a single vertex.

Definition 2.1. Let G be a cubic graph and $w : E(G) \to \{1, 2\}$ be a weight of G. A family $\mathcal{F}$ of a circuits of G is a *faithful circuit cover of the weighted graph* (G, w) if every edge e is contained in precisely $w(e)$ members of $\mathcal{F}$. A weight w is *eulerian* if the total weight of every edge-cut is even.

Let w be an eulerian weight of G. The set of edges with weight i is denoted by $E_{w=i}$. (G, w) is a $(1, 2)$-eulerian weighted graph if $w(e) = 1$ or 2 for every $e \in E(G)$.

An example of a faithful circuit cover is illustrated in Fig. 3.

It is obvious that *bridgeless* and *eulerian* are necessary conditions for a graph G to have a faithful circuit cover with respect to w. The circuit double cover conjecture is obviously a special case of the faithful circuit cover problem where the weight w is 2 for every edge.

Unfortunately, not every eulerian weighted graph has a faithful cover, for example, (P_{10}, w_{10}) (see Fig. 4).

Definition 2.2. *A contra pair* (G, w) is an eulerian weighted graph that does not have a faithful circuit cover.

Definition 2.3. Let $\mathcal{C} = \{C_1, \cdots, C_s\}$ be a set of circuits of a graph G. The eulerian weight $w_{\mathcal{C}}$ of G induced by the coverage of $\mathcal{C}$ is defined as follows:

$$w_{\mathcal{C}}(e) = |\{C \in \mathcal{C} : \ e \in C\}|.$$

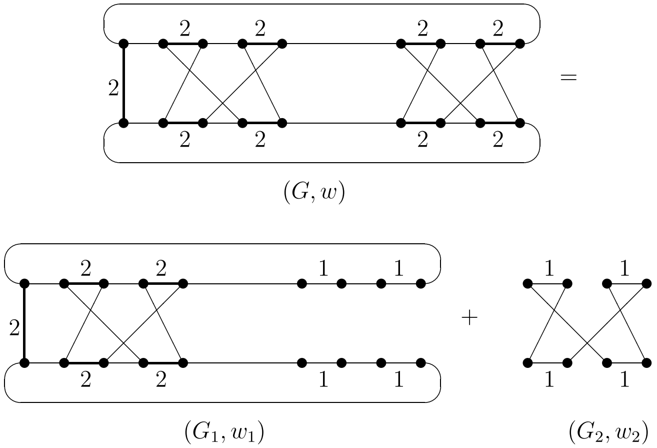

Fig. 5. Weight decomposition $(G, w) = (G_1, w_1) + (G_2, w_2)$ where G_2 is a removable circuit.

It is obvious that $w_{\mathcal{C}}$ is eulerian since $\mathcal{C}$ is a set of circuits.

Let (G, w) be a $(1, 2)$-eulerian weighted graph and $e_0 \in E_{w=2}$ such that $G - e_0$ is bridgeless. With no confusion and a slight abuse of notation, $(G - e_0, w)$ is the weighted graph where w is the restriction of w on the edge set $E(G) - \{e_0\}$. Since w is eulerian and $w(e_0) = 2$, the weight w restricted on the edge set $E(G) - \{e_0\}$ remains eulerian. Thus, two edges incident with an endvertex of e_0 must have the same weight. Hence, the weighted suppressed graph $(\overline{G - e_0}, w)$ is well-defined.

A *removable circuit*, which is a very natural concept in an inductive approach for circuit covering problems, will be defined after the following general definition.

Definition 2.4. Let (G, w) be a $(1, 2)$-eulerian weighted graph. A w-decomposition of (G, w) is a pair of $(1, 2)$-eulerian weighted graphs $\{(H_1, w_1), (H_2, w_2)\}$ where H_1 and H_2 are subgraphs of G with $H_1 \cup H_2 = G$ and w_i is an eulerian weight of H_i $(i = 1, 2)$ such that

$$w(e) = \begin{cases} w_1(e) & \text{if } e \in H_1 - H_2 \\ w_2(e) & \text{if } e \in H_2 - H_1 \\ w_1(e) + w_2(e) & \text{if } e \in H_1 \cap H_2. \end{cases}$$

(See Fig. 5.)

Definition 2.5. Let (G, w) be a $(1, 2)$-eulerian weighted graph and let $\{(H_1, w_1), (H_2, w_2)\}$ be a w-decomposition of (G, w) such that H_1 is a circuit with $w_1 \equiv 1$. If H_2 is bridgeless, then H_1 is called a *removable circuit* of (G, w). (See Fig. 5.)

Definition 2.6. A contra pair (G, w) is *minimal* if, for every $e \in E_{w=2}$, the weighted graph $(G - e, w)$ has a faithful circuit cover, and (G, w) has no removable circuit.

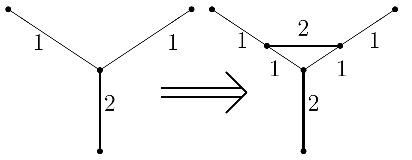

Fig. 6. ($Y \to \triangle$) operation.

Definition 2.7. Let G be a cubic graph and H_1, H_2 be subgraphs of G. An *attachment of* H_2 *in the suppressed graph* $\overline{H_1}$ is an edge $e = uv$ of $\overline{H_1}$ such that the edge e corresponds to a maximal induced path $P = u \cdots v$ (in H_1) and $V(H_2) \cap [V(P) - \{u, v\}] \neq \emptyset$.

3. Hamilton weight

As we mentioned above, in order to study the structure of circuit chains and make possible adjustments of circuit covers, one of the most basic and natural steps is the characterization of the subgraph induced by two incident circuits. This motivates us to study $(1, 2)$-eulerian weighted graphs with precisely two Hamilton circuits as a faithful cover.

Definition 3.1 *(Hamilton weight).* Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \to \{1, 2\}$. If the eulerian weighted graph (G, w) has a faithful circuit cover and every faithful circuit cover of (G, w) is a pair of Hamilton circuits, then w is a *Hamilton weight* of G, and (G, w) is called a *Hamilton weighted graph*.

Definition 3.2 *(($Y \to \triangle$)-operation).* (See Fig. 6.) Let v be a degree 3 vertex of an eulerian weighted graph (G, w) incident with $E(v) = \{e_i = vu_i : \ i = 1, 2, 3\}$. *A* $(Y \to \triangle)$*-operation* of (G, w) at the vertex v is the construction of a new eulerian weighted graph (G', w') from (G, w) by splitting v to be three degree 1 vertices $\{v_1, v_2, v_3\}$ where v_i is incident with e_i, and adding a triangle $v_1v_2v_3v_1$ and assigning $w'(v_jv_i) = w'(v_hu_h) = w(vu_h)$ to new edges for every $\{h, i, j\} = \{1, 2, 3\}$.

Observation 3.3. *Let* (G', w') *be a weighted graph obtained from another weighted graph* (G, w) *via a* $(Y \to \triangle)$*-operation. Then* w *is a Hamilton weight of* G *if and only if* w' *is a Hamilton weight of* G'*.*

The weighted graph $(3K_2, w_2)$ consists of two vertices and three parallel edges such that one edge is of weight 2 while other two are of weights 1.

Definition 3.4. The family of weighted graphs constructed from $(3K_2, w_2)$ via a series of $(Y \to \triangle)$-operations is denoted by $\langle \mathcal{K}_4 \rangle$.

The three smallest cubic graphs in $\langle \mathcal{K}_4 \rangle$ are illustrated in Fig. 7. A Hamilton weight is also illustrated in the figure: double lines are weight 2 edges, and single lines are weight

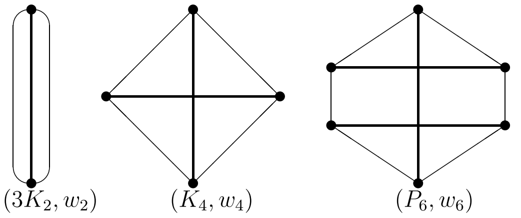

Fig. 7. Three small $\langle \mathcal{K}_4 \rangle$-graphs.

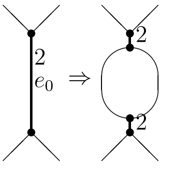

Fig. 8. Edge-dividing operation.

1 edges. It is easy to see that, for every $(G, w) \in \langle \mathcal{K}_4 \rangle$, $E_{w=1}$ induces a 2-factor while $E_{w=2}$ induces a 1-factor.

The Hamilton Weight Conjecture. *(See [21].) Let (G, w) be a Hamilton weighted graph. If (G, w) is 3-connected, then $(G, w) \in \langle \mathcal{K}_4 \rangle$.*

This conjecture was proved for the family of *Petersen-minor free* graphs [14] and its relation with the *unique* 3-*edge-coloring* problem can be found in [21,22].

Definition 3.5 *(Edge-dividing).* (See Fig. 8.) Let (G, w) be an eulerian weighted graph and $e_0 \in E_{w=2}$ with end-vertices x_1 and x_2. Let G^* be the cubic graph obtained from G by deleting the edge e_0 and adding two new vertices $\{y_1, y_2\}$ and four new edges $\{e_1, e_2, f_1, f_2\}$ where, for each $i = 1, 2$, x_i and y_i are the endvertices of e_i, and, y_1 and y_2 are the endvertices of f_i. Let w^* be the weight of G^* obtained from w: $w^*(e) = w(e)$ if $e \notin \{e_1, e_2, f_1, f_2\}$, and $w^*(e_i) = 2$, $w^*(f_i) = 1$ for each $i = 1, 2$. Then (G^*, w^*) is the weighted graph obtained from (G, w) via an *edge-dividing operation* at e_0.

Observation 3.6. *Let (G', w') be a weighted graph obtained from another weighted graph (G, w) via an edge-dividing operation. Then w is a Hamilton weight of G if and only if w' is a Hamilton weight of G'.*

Definition 3.7. The family of weighted graphs (G, w) constructed from $(3K_2, w_2)$ via a series of operations, each of which is either a $(Y \to \triangle)$-operation or an edge-dividing operation, is denoted by $\langle \mathcal{K}_4 \rangle_2$.

Observation 3.8. *If $(G, w) \in \langle \mathcal{K}_4 \rangle_2$ other than $(3K_2, w_2), (K_4, w_4)$, then (G, w) has a pair of disjoint small circuits C_1, C_2, each of which is either a digon with total weight 2 or a triangle with total weight 4.*

Lemma 3.9. *(See Lemma 3.3 in [23].) Let $(G; w)$ be a Hamilton weighted graph. Then the total weight of every edge cut of G is at least 4.*

Lemma 3.10. *Under the assumption of the Hamilton weight conjecture, every Hamilton weighted graph is a member of $\langle \mathcal{K}_4 \rangle_2$.*

Proof. Induction on $|E(G)|$. It is trivial if G is 3-connected. Hence, assume that $T = \{e_1, e_2\}$ is a 2-edge-cut with components Q_1, Q_2. Without loss of generality, let $|E(Q_1)| \geq |E(Q_2)|$.

Let $G_1 = \overline{G/Q_2}$ and w_1 be the resulting weight. It is trivial that (G_1, w_1) is smaller than (G, w) and is a Hamilton weighted graph. Hence, by induction, $(G_1, w_1) \in \langle \mathcal{K}_4 \rangle_2$. Let f_1 be the weight 2 edge of (G_1, w_1) created by the contraction of Q_2 and the suppression of the resulting degree 2 vertex. (Note that, by Lemma 3.9, the edge f_1 is of weight 2.)

The lemma is trivial if $(G_1, w_1) = (3K_2, w_2)$ or (K_4, w_4) (note that $|E(Q_1)| \geq |E(Q_2)|$). Thus, by Observation 3.8, let $\{C_1, C_2\}$ be the pair of disjoint small circuits in (G_1, w_1) described in Observation 3.8. Without loss of generality, let $f_1 \notin E(C_1)$. It is evident that $(\overline{G/C_1}, w)$ is also a Hamilton weighted graph, and, therefore, by induction, it is a member of $\langle \mathcal{K}_4 \rangle_2$. Thus, the Hamilton weighted graph (G, w) is constructed from $(\overline{G/C_1}, w)$, a member of $\langle \mathcal{K}_4 \rangle_2$, via a $(Y \to \triangle)$-operation if $|E(C_1)| = 3$ or an edge-dividing operation if $|E(C_1)| = 2$. $\square$

4. Circuit chain plus an edge (CCPE graph), Petersen chain and the main theorem

Recall that a circuit chain joining vertices x_0, y_0 is a family of circuits $C_1, \cdots, C_t$ with $x_0 \in V(C_1) - V(C_2)$, $y_0 \in V(C_t) - V(C_{t-1})$ and $V(C_i) \cap V(C_j) \neq \emptyset$ if and only if $i = j \pm 1$. (See Fig. 1.)

Definition 4.1. Let G be a bridgeless cubic graph with an eulerian weight $w : E(G) \to \{1, 2\}$, and let $e_0 = x_0y_0$ be a weight 2 edge. The eulerian weighted graph (G, w) is called *a circuit chain plus an edge e_0* (abbreviated as *CCPE*-graph, see Fig. 9), if $(G - e_0, w)$ has a faithful circuit cover $\{C_1, C_2, \cdots, C_t\}$ that forms a circuit chain connecting the vertices x_0 and y_0.

Definition 4.2. Let G be a bridgeless cubic graph with an eulerian weight $w : E(G) \to \{1, 2\}$, and let $e_0 = x_0y_0$ be a weight 2 edge. The eulerian weighted graph (G, w) is called *a simple Petersen chain with a bowstring e_0* (see Fig. 9) if (G, w) has a set of minimal 3-edge-cuts $\{T_1, T_2, \cdots, T_c\}$ such that

(1) $e_0 \in T_\mu$ and $w(T_\mu) = 4$ for every $\mu = 1, \cdots, c$; and $T_i \cap T_j = \{e_0\}$ if $i \neq j$;

(2) let Q_μ, R_μ be components of $G - T_\mu$ with $x_0 \in Q_\mu$ and $y_0 \in R_\mu$,

$$\begin{array}{ccccccccccc} \{x_0\} & = & Q_1 & \subset & Q_2 & \subset & \cdots & \subset & Q_c & = & G - \{y_0\}, \\ G - \{x_0\} & = & R_1 & \supset & R_2 & \supset & \cdots & \supset & R_c & = & \{y_0\}; \end{array}$$

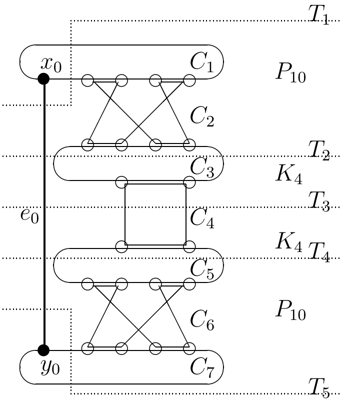

Fig. 9. A circuit chain joined by e_0: Petersen chain.

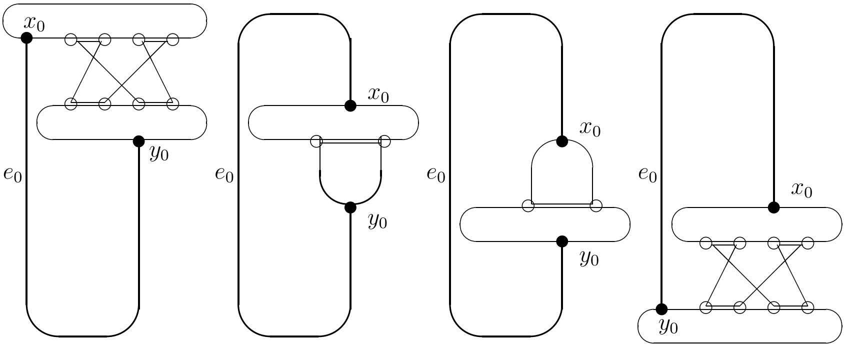

Fig. 10. Segments of the Petersen chain in Fig. 9.

(3) for each $\mu = 1, \cdots, c-1$, the contracted weighted graph $(S_\mu = G/[Q_\mu \cup R_{\mu+1}], w)$ is either (P_{10}, w_{10}) or (K_4, w_4) (the contracted graph S_μ is called a *segment* of the chain-see Fig. 10).

Note that, by the structure of (K_4, w_4) and (P_{10}, w_{10}), the set of 3-edge-cuts $\{T_1, \cdots, T_c\}$ consists of all minimal 3-edge-cuts containing e_0 of a simple Petersen chain.

Definition 4.3. *A Petersen chain* (G, w) *with a bowstring* e_0 is obtained from a simple Petersen chain with the bowstring $e_0 = x_0 y_0$ via a series of operations, each of which is either a $(Y \to \triangle)$-operation at any vertex other than x_0 and y_0, or an edge-dividing operation at any weight 2 edge other than e_0.

Since these $(Y \to \triangle)$-operations and edge-dividing operations do not create any new 2- or 3-edge-cut containing e_0, the set $\{T_1, \cdots, T_c\}$ (described in Definition 4.2) remains the set of all minimal 3-edge-cuts containing e_0 of a Petersen chain. Hence, we have the following observation.

Observation 4.4. *Let G be a Petersen chain with a bowstring e_0, and let $\{T_1, T_2, \cdots, T_c\}$ be the set of all minimal 3-edge-cuts containing the edge e_0. Then*

(1) $w(T_\mu) = 4$ for every $\mu = 1, \cdots, c$; and $T_i \cap T_j = \{e_0\}$ if $i \neq j$;

(2) let Q_μ, R_μ be components of $G - T_\mu$ with $x_0 \in Q_\mu$ and $y_0 \in R_\mu$,

$$\begin{array}{ccccccccccc} \{x_0\} & = & Q_1 & \subset & Q_2 & \subset & \cdots & \subset & Q_c & = & G - \{y_0\}, \\ G - \{x_0\} & = & R_1 & \supset & R_2 & \supset & \cdots & \supset & R_c & = & \{y_0\}. \end{array}$$

For each $\mu = 1, \cdots, c-1$, let $(S_\mu = G/[Q_\mu \cup R_{\mu+1}], w)$ be the contracted weighted graph. Then either (P_{10}, w_{10}) or (K_4, w_4) can be obtained from $(S_\mu = G/[Q_\mu \cup R_{\mu+1}], w)$ by recursively contracting triangles and digons not containing x_0, y_0, and recursively suppressing resulting degree 2 vertices. (Each S_μ is called a segment *of the chain, see Fig. 10.)*

Definition 4.5. A segment S_μ of a Petersen chain with bowstring $e_0 = x_0y_0$ is called a K_4-segment (or P_{10}-segment, respectively) if K_4 (or P_{10}, respectively) can be obtained from S_μ be recursively contracting triangles/digons and recursively suppressing degree 2-vertices (see Fig. 10).

Definition 4.6. A single segment Petersen chain, as the name indicates, is a Petersen chain with precisely one segment.

The following theorem is the main theorem which characterizes the structure of CCPE graphs.

Theorem 4.7. *Let G be a bridgeless cubic graph with an eulerian weight $w : E(G) \to \{1, 2\}$, and let $e_0 = x_0y_0$ be a weight 2 edge. Assume that $(G - e_0, w)$ has a faithful circuit cover $\{C_1, C_2, \cdots, C_t\}$ that forms a circuit chain connecting the vertices x_0 and y_0 and (G, w) has no removable circuit C with $e_0 \notin E(C)$, then, under the assumption of the Hamilton weight conjecture, (G, w) is a Petersen chain with e_0 as the bowstring edge.*

4.1. Extension from earlier results

Let G be a bridgeless cubic graph with a $(1, 2)$-eulerian weight w. We further assume that *(G, w) is a contra pair.*

In the paper [23], it was proved that

(*) If, for every $e_0 = x_0y_0 \in E_{w=2}$, $(G - e_0, w)$ has a faithful circuit cover which is a circuit chain joining x_0 and y_0 and (G, w) does not have any removable circuit avoiding e_0, then $(G, w) = (P_{10}, w_{10})$ (under the assumption of the Hamilton weight conjecture).

The result (*) is further strengthened in this paper as follows (an equivalent version of Theorem 4.7),

(**) If, for <u>*a given*</u> $e_0 = x_0y_0 \in E_{w=2}$*,* $(G - e_0, w)$ *has a faithful circuit cover which is a circuit chain joining* x_0 *and* y_0 *and* (G, w) *does not have any removable circuit avoiding* e_0*, then* (G, w) *is a Petersen chain (under the assumption of the Hamilton weight conjecture).*

The following is a brief discussion about the difference between these two results and their proofs.

I. We can show that (*) is a corollary of (**): Arguments from [23] show that a contra pair for (*) does not have a nontrivial edge cut of size at most 3, so by (**) we must have a single segment Petersen chain without digons or triangles.

II. The result (*) is motivated by a long-standing open problem (Fleischner and Jackson [7]) that *every minimal contra pair must be the weighted Petersen graph* (P_{10}, w_{10}).

III. However, the main result (**) of this paper is mainly for the characterization of circuit chain structure (Question 1.1) and its future applications, such as, the determination of local structure of contra pairs, or adjustment of a circuit chain. In order to have some useful lemmas for future applications, a missing weight 2 edge e_0 (that links the ends of the chain) must be a <u>given</u> edge. That is, the existence of a faithful cover for $(G - e, w)$ holds for that <u>given</u> edge $e = e_0$, but may not hold for <u>other</u> weight 2 edges e.

IV. The proof of (*) in [23] relies on a structural result in [2] that *a weighted graph described in (*) must be a permutation graph*. However, this structure cannot be applied any more in this paper because of the difference of "$\forall$" or "$\exists$" $e_0 \in E_{w=2}$. This makes the proof (Section 6) much more complicated: without knowing the structure as a permutation graph, we have to go through a completely new (and lengthy) proof for finding a removable circuit (Subsection 6.3).

On the other hand, as we shall also point out, the very detailed description of the Petersen chain does provide a certain advantage in the inductive proof; it makes part of the proof relatively shorter (such as: Claim 10 in Subsection 6.1): in the induction proof, we are able to use the well-described structure of a sub-chain which was not available at all in [23].

5. Lemmas

5.1. Faithful covers

Theorem 5.1. *(See Alspach and Zhang [1], or see [2].) Let G be a bridgeless cubic graph. If G contains no Petersen graph subdivision, then G has a faithful circuit cover with respect to every* $(1, 2)$*-eulerian weight.*

5.2. Observations about $\langle \mathcal{K}_4 \rangle$*-graphs*

Observation 5.2. *For each* $(G, w) \in \langle \mathcal{K}_4 \rangle_2$ *with* $|V(G)| \geq 6$*, we have the following properties.*

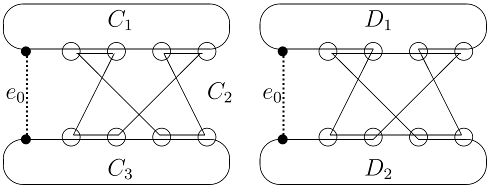

Fig. 11. Circuit chains $\mathcal{F}_1 = \{C_1, C_2, C_3\}$ and $\mathcal{F}_2 = \{D_1, D_2\}$ in $P_{10} - e_0$.

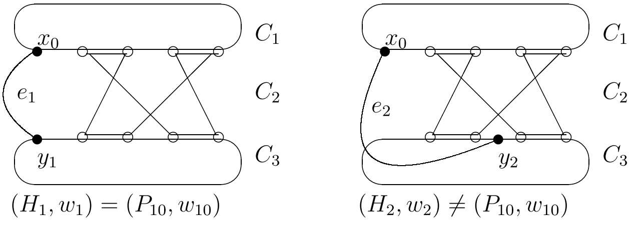

Fig. 12. One is P_{10}, while another is not.

(1) (G, w) has a non-trivial 2- or 3-edge-cut with total weight 4; and $(G, w) \in \langle \mathcal{K}_4 \rangle$ if and only if G is 3-connected.

(2) For each non-trivial 2- or 3-edge-cut T with components Q_1 and Q_2, each Q_j contains a triangle with total weight 4 or a digon with total weight 2.

(3) All triangles of (G, w) are vertex-disjoint if $(G, w) \in \langle \mathcal{K}_4 \rangle$.

Lemma 5.3. *(See Lemma 6.3 in [23].) Let S be a triangle of a weighted graph (G, w). Then $(G, w) \in \langle \mathcal{K}_4 \rangle$ if and only if the contracted weighted graph $(G/S, w) \in \langle \mathcal{K}_4 \rangle$.*

5.3. Observations about Petersen graph and Petersen chain

Proposition 5.4. *The weighted graph $(K_4 - e_0, w_4)$ has precisely one faithful circuit cover for each $e_0 \in E_{w_4=2}$. The weighted graph $(P_{10} - e_0, w_{10})$ has precisely two faithful circuit covers $\mathcal{F}_1, \mathcal{F}_2$ for each $e_0 \in E_{w_{10}=2}$ where $|\mathcal{F}_1| = 3$ and $|\mathcal{F}_2| = 2$. (See Fig. 11.)*

Proposition 5.5. *Let G be a graph with 11 vertices: $d(v_i) = 2$ for $i = 0, 1, 2$ and $d(v_j) = 3$ for $j = 3, \cdots, 10$. Construct a new graph G_i from G by adding a new edge joining v_0 and v_i for each $i = 1, 2$. If v_1 and v_2 are not adjacent, then at most one of $\{\overline{G_1}, \overline{G_2}\}$ is isomorphic to the Petersen graph. (See Fig. 12.)*

V_8 is the cubic graph consisting of a Hamilton circuit $v_0 v_1 \cdots v_7 v_0$ and a perfect matching $\{v_i v_{i+4} : i = 0, 1, 2, 3\}$. Since $\overline{P_{10} - e} = V_8$ for any $e \in E(P_{10})$, and $\overline{P_{10} - v} = K_{3,3}$ for any $v \in V(P_{10})$, we have the following observation.

Proposition 5.6. *For any edge $e \in E(P_{10})$ or any vertex $v \in V(P_{10})$, $P_{10} - e$ and $P_{10} - v$ remain non-planar.*

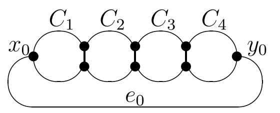

Fig. 13. Circuit chain joining x_0 and y_0.

The following is a straightforward observation from the definition of Petersen chain and Proposition 5.4.

Lemma 5.7. *Let (G, w) be a Petersen chain with a bowstring $e_0 = x_0y_0$. If $\mathcal{P} = \{C_1, \cdots, C_t\}$ is a circuit chain of (G, w) joining x_0, y_0 with $|\mathcal{P}| = t$ maximum, then each minimal edge-cut T_i of size 3 ($i = 1, \cdots, c$) containing e_0 must also contain two weight one edges of $C_{\phi(i)}$ where $\phi : \{1, \cdots, c\} \to \{1, \cdots, t\}$ is a one-to-one mapping such that*

(1) $1 = \phi(1) < \phi(2) < \cdots < \phi(c) = t$;

(2) $\phi(\mu + 1) - \phi(\mu) = 1$ or 2;

(3) If $\phi(\mu + 1) - \phi(\mu) = 1$, then the segment S_μ is a K_4-segment;

(4) If $\phi(\mu + 1) - \phi(\mu) = 2$ then the segment S_μ is a P_{10}-segment.

Note that K_4- and P_{10}-segments are defined in Definition 4.5.

5.4. Circuit chain with faithful cover

The following lemma is useful and will be applied to solve some special cases for Theorem 4.7 and some other useful lemmas for further applications.

Lemma 5.8. *Let (G, w) be a CCPE graph consisting of a circuit chain $\mathcal{P} = \{C_1, \cdots, C_t\}$ plus a weight 2 edge $e_0 = x_0y_0$ such that (G, w) has no removable circuit C with $e_0 \notin E(C)$ (the same description as in Theorem 4.7). If the eulerian weighted graph (G, w) itself has a faithful circuit cover, then, under the assumption of the Hamilton weight conjecture,*

(1) $(G, w) \in \langle \mathcal{K}_4 \rangle_2$;

(2) $|E(C_\mu) \cap E(C_{\mu+1})| = 1$ for every $\mu = 1, \cdots, t - 1$ if every triangle and digon of G contains either x_0 or y_0 (see Fig. 13). That is, (G, w) is a Petersen chain with e_0 as the bowstring and every segment of the Petersen chain is a K_4-segment.

Proof. It is obvious that every faithful cover of (G, w) consists of precisely two circuits, for otherwise, the third one not containing e_0 is removable. So, w is a Hamilton weight of G and, therefore, by Lemma 3.10, $(G, w) \in \langle \mathcal{K}_4 \rangle_2$. This proves the conclusion (1).

Let (G, w) be a smallest counterexample to the conclusion (2) of the lemma. It is easy to see that $|V(G)| \geq 6$.

Consider a non-trivial 2- or 3-edge-cut T separating G into components Q_1 and Q_2. By Observation 5.2-(1) such a T exists. By Observation 5.2-(2) each Q_i contains a triangle or digon, so x_0 is in Q_1 and y_0 is in Q_2 (or vice versa) and $e_0 \in T$. The edges incident with every digon or triangle form a 2- or 3-edge-cut T, so e_0 is incident with every digon or triangle. Hence G contains exactly two circuits of length 2 or 3, one containing x_0 and the other y_0. If either is a digon then there is a 2-edge-cut containing e_0, contradicting the fact that $\{C_1, \cdots, C_t\}$ is a circuit chain from x_0 to y_0. Therefore there are exactly two triangles, which must be C_1 and C_t.

Since $(G/C_t, w)$ is smaller than the smallest counterexample, conclusion (2) holds for $(G/C_t, w)$. That is, $|E(C_\mu) \cap E(C_{\mu+1})| = 1$ for each $\mu = 1, \cdots, t-2$. The proof of (2) is completed since C_t is a triangle that intersects C_{t-1} with precisely one edge. $\square$

Lemma 5.9. *Let (G, w) be a CCPE graph consisting of a circuit chain $\mathcal{P} = \{C_1, \cdots, C_t\}$ plus a weight 2 edge $e_0 = x_0y_0$ such that (G, w) has no removable circuit C with $e_0 \notin E(C)$ (the same description as in Theorem 4.7). Assume that $|\mathcal{P}| = t$ is maximum. Let f_{x_0}, f_{y_0} be subdivided edges of $G - e_0$ containing x_0 or y_0, respectively. If $|\mathcal{P}| = t = 2$, then, under the assumption of the Hamilton weight conjecture,*

(1) $(G, w) \in \langle \mathcal{K}_4 \rangle_2$;

(2) there is a 3-edge-cut of $\overline{G - e_0}$ containing both subdivided weight one edges f_{x0}, f_{y0};

(3) every 3-edge-cut of G containing e_0 is trivial (that is, $E(x_0)$ and $E(y_0)$ are the only two 3-edge-cuts of G containing e_0).

Proof. By Lemma 3.10 and the choice of $\mathcal{P}$ that $|\mathcal{P}| = 2$ is maximum, the weighted graph $(\overline{G - e_0}, w) \in \langle \mathcal{K}_4 \rangle_2$. Since every member of $\langle \mathcal{K}_4 \rangle_2$ is planar, by Proposition 5.6, G does not contain a subdivision of the Petersen graph. Hence, by Theorem 5.1, (G, w) has a faithful cover. Conclusion (1) of the lemma follows immediately from Lemma 5.8-(1).

Now we only need to prove the conclusions (2) and (3).

After recursively contracting all triangles/digons not containing x_0, y_0 and recursively suppressing all degree 2 vertices (along some subdivided weight 2 edges), we still satisfy the conditions of Theorem 4.7 with $t = 2$, and we still have a faithful circuit cover, so by Lemma 5.8-(2), we have (K_4, w_4). The lemma holds for (K_4, w_4), and so holds for (G, w) (since none of those operations (or their inverses) affects the conclusions (2) and (3) of the lemma). $\square$

5.5. L-graphs

Before the proof of the main theorem (Theorem 4.7), we introduce a new concept, L-graph, which is critical in the final determination of the Petersen graph structure.

Definition 5.10. A weighted L-graph is a cubic graph L of order $2n$ ($n \geq 2$) associated with an eulerian weight $w : E(L) \to \{1, 2\}$ and a weight one edge $e_0 = v_0v_n$ (called *a diagonal crossing chord*) such that

(1) $(L, w) \in \langle \mathcal{K}_4 \rangle$,

(2) every triangle of L must contain either v_0 or v_n.

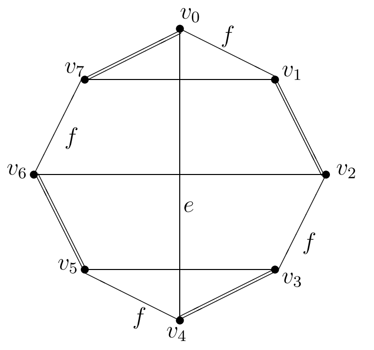

Fig. 14. An L-graph with the diagonal crossing chord e and F.

(See Fig. 14; we will show that all L-graphs have a similar structure.)

Lemma 5.11. *The following two statements are equivalent:*

(1) (L, w) *is an L-graph with a diagonal crossing chord* $e^* = x^*y^*$*;*

(2) Let e^* *be a weight one edge of* $(3K_2, w_2)$*,* (L, w) *is constructed recursively from* $(3K_2, w_2)$ *by a series of* $(Y \to \triangle)$*-operations* only *at some endvertex of* e^**. (Note that the edge* e^* *will remain as the diagonal crossing chord during the expansion of the L-graph.)*

Proof. (2) $\Rightarrow$ (1) is trivial. We prove (1) $\Rightarrow$ (2) by induction on $|V(L)|$. The lemma is true if $|V(L)| \leq 4$. So, by Observation 5.2-(3), L has precisely two triangles, each contains precisely one of $\{x^*, y^*\}$. Let S be a triangle of L containing x^* (but not y^*). By Lemma 5.3, $(L/S, w) \in \langle \mathcal{K}_4 \rangle$ and, without causing any confusion, denote the new contracted vertex by x^* which remains as an endvertex of e^*. It is easy to see that $(L/S, w)$ is an L-graph (by Definition 5.10). Since any resulting triangle (after contraction of S) must contain the contracted vertex x^*, by induction, (1) $\Rightarrow$ (2) for $(L/S, w)$. Now (1), and hence (2), is true for $(L/S, w)$, so (2) holds for (L, w), since (L, w) is obtained from $(L/S, w)$ via a $(Y \to \triangle)$-operation at x^*. $\square$

Since an L-graph $(L, w) \in \langle \mathcal{K}_4 \rangle$ and is a Hamilton weighted graph, let $\{C_1, C_2\}$ be the faithful circuit cover of (L, w). Each C_j is a Hamilton circuit. One may draw the L-graph (L, w) on the plane as follows (by Lemma 5.11, see Fig. 14):

The Hamilton circuit $C_2 = v_0 \cdots, v_{2n-1} v_0$ is the boundary of the exterior face with a *diagonal crossing chord* $v_0 v_n$ and a set Z of *parallel chords* where $Z = \{v_{2n-\mu} v_\mu : \mu = 1, \cdots, n-1\}$. And another Hamilton circuit $C_1 = v_0 v_{2n-1} v_1 v_2 v_{2n-2} v_{2n-3} v_3 v_4 \cdots v_n v_0$, and w is a Hamilton weight with $E_{w=2} = \{v_{2i-1} v_{2i} : i = 1, \cdots, n\}$ where C_1 and C_2 intersect.

Fig. 14 is an illustration of a weighted L-graph with 8 vertices. Note that, in Fig. 14, double lines are edges in $E_{w=2}$ and single lines are edges in $E_{w=1}$.

One can see that all parallel chords (Z-chords) do not cross each other, while the diagonal crossing chord $v_0 v_n$ crosses every parallel chord.

Lemma 5.12. *Let $(L, w) \in \langle \mathcal{K}_4 \rangle$ of order $2n$ (≥ 4). Let $\{C_1, C_2\}$ be a faithful circuit cover of (L, w). Let $e \in C_1 - C_2$ and $F \subseteq C_2 - C_1$. Assume that*

(a) every triangle of L contains some edge of $F \cup \{e\}$ and,

(b) for every edge $f \in F$, L contains a 3-edge-cut T with both $f, e \in T$.

Then (L, w) must be a weighted L-graph described above with $e = v_0 v_n$ as the diagonal crossing chord and

$$\{v_0 v_1, v_\nu v_{\nu+1}\} \subseteq F \subseteq \{v_{2i} v_{2i+1} : i = 0, \cdots, n-1\} = E(C_2) - E(C_1)$$

where $\nu = n$ if n is even and $\nu = n - 1$ if n is odd.

Proof. By Definition 5.10, we only need to show that every triangle of L must contain an endvertex of $e = v_0 v_n$.

Suppose that S is a triangle of L such that v_0 and $v_n \notin V(S)$ (and so $e \notin E(S)$). Hence, by (a), let $f = z_1 z_2 \in E(S) \cap F$. By (b), there is a 3-edge-cut T containing both e and f.

Note that $|S \cap T|$ must be even since one is a circuit, while another one is a cut. Since $f \in S \cap T$, $|S \cap T| = 2$. Note that S is a triangle, so either $T = E(z_j)$ (for some $j \in \{1, 2\}$) or L has a 2-edge-cut $E(z_i) \triangle T$ (for some $i \in \{1, 2\}$). So, $T = E(z_j)$ since L is 3-connected. Hence, both $f, e \in E(z_j)$ for some $j \in \{1, 2\}$. This contradicts that v_0 and $v_n \notin V(S)$. □

Fig. 14 is an illustration of a weighted L-graph with 8 vertices in which the circuit $C_2 = v_0 \cdots v_7 v_0$ and the circuit $C_1 = v_0 v_7 v_1 v_2 v_6 v_5 v_3 v_4 v_0$ and edges labeled with f are possible locations of edges of F.

6. Proof of Theorem 4.7

6.1. First part of the proof: the case of $|\mathcal{P}| > 3$

Let (G, w) be a smallest counterexample to the theorem. And we choose $t = |\mathcal{P}|$ as large as possible.

I. Since (G, w) has no removable circuit avoiding e_0, we have the following claim for (G, w).

Claim 1. *Every faithful circuit cover of $(G - e_0, w)$ is a circuit chain joining x_0 and y_0.*

By Lemma 5.9, if $t = 2$, then $(G, w) \in \langle \mathcal{K}_4 \rangle_2$ (a single segment Petersen chain). It contradicts that (G, w) is a counterexample. Hence,

Claim 2. $t \geq 3$.

By Lemma 5.8, we have that

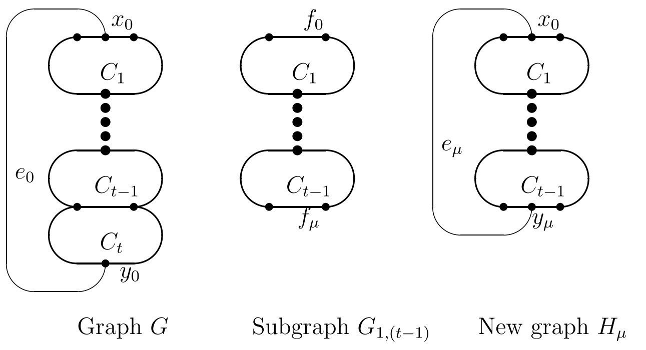

Fig. 15. Circuit chain and subchain.

Claim 3. *(G, w) is a contra pair.*

Since $t \geq 3$ there are no circuits of length ≤ 3 containing e_0. Therefore any circuit of length ≤ 3 can be contracted to obtain a smaller CCPE graph, which is a Petersen chain by Theorem 4.6; then (G, w) is also a Petersen chain, a contradiction. Hence,

Claim 4. *G is of girth at least 4.*

Claim 5. *G does not contain any non-trivial 3-edge-cut T consisting of e_0 and a pair of weight one edges.*

Proof. For otherwise, let Q and R be the components of $G - T$, one may apply the theorem to the smaller CCPE graphs $(G/Q, w)$ and $(G/R, w)$. □

II.

Notation 6.1. *For $1 \leq \alpha < \beta \leq t$, let $(G_{\alpha,\beta}, w_{\alpha,\beta})$ be the induced subgraph $G[C_\alpha \cup \cdots \cup C_\beta]$ associated with the eulerian weight $w_{\{C_\alpha,\cdots,C_\beta\}}$ induced by the circuit subchain $\{C_\alpha, \cdots, C_\beta\}$. (See Fig. 15. See Definition 2.3 for induced eulerian weight $w_{\{C_\alpha,\cdots,C_\beta\}}$.)*

Claim 6. *For each $\mu < t$, the number of attachments of $C_{\mu+1}$ in $(\overline{G_{1,\mu}}, w_{1,\mu})$ is at least 2.*

Proof. For otherwise, G has a 3-edge-cut consisting of e_0 and two weight ones edges of C_μ (part of the attachment of $C_{\mu+1}$ in $(\overline{G_{1,\mu}}, w_{1,\mu})$). This contradicts Claim 5. □

By Lemma 3.10 and the assumption that $|\mathcal{P}|$ is maximum.

Claim 7. $(\overline{G_{\mu,(\mu+1)}}, w_{\mu,(\mu+1)}) \in \langle \mathcal{K}_4 \rangle_2$.

Claim 8. *G does not have any 2-edge-cut T separating e_0 from other edges.*

Proof. Suppose that T is a 2-edge-cut of G with components Q' and Q'' and $e_0 \in Q'$.

If $w(T) = 2$ then only one circuit C_μ of $\mathcal{P}$ passes through T, which means Q'' contains only vertices of C_μ, which is impossible.

So $w(T) = 4$ and two circuits $C_\mu, C_{\mu+1}$ pass through T. Since (G, w) has no removable circuit avoiding e_0, $\{C_\mu, C_{\mu+1}\}$ covers Q''. Hence, Q'' is a subgraph of $G_{\mu,(\mu+1)}$. By Claim 7 and Observation 5.2, Q'' contains a triangle or digon. This contradicts Claim 4. $\square$

Claim 9. *For each $i = 1, \cdots, t-1$, the suppressed cubic graph $\overline{G_{i,i+1}}$ is 3-connected and, therefore, the weighted graph $(\overline{G_{i,i+1}}, w_{i,i+1}) \in \langle \mathcal{K}_4 \rangle$.*

Proof. By Claim 7, $(\overline{G_{i,i+1}}, w_{i,i+1})) \in \langle \mathcal{K}_4 \rangle_2$. Let $(J, w_J) = (\overline{G_{i,i+1}}, w_{i,i+1}))$.

Suppose that $(J, w_J) \in \langle \mathcal{K}_4 \rangle_2 - \langle \mathcal{K}_4 \rangle$. By Observation 5.2-(1), J has a 2-edge-cut T with $w(T) = 4$ and with components Q' and Q''. By Claim 8, let Q' contain an attachment z' of C_{i-1} (or contain the vertex x_0 if $i = 1$), and Q'' contain an attachment z'' of C_{i+2} (or contain the vertex y_0 if $i + 1 = t$). Let D be the component of $E_{w_J=1}$ containing z''. Then, $\{C_i \triangle D, C_{i+1} \triangle D\}$ is another faithful cover of (J, w_J) consisting of precisely two circuits (since $|\mathcal{P}|$ is maximum). Hence, $\mathcal{P} - \{C_i, C_{i+1}\} + \{C_i \triangle D, C_{i+1} \triangle D\}$ is a faithful cover of $(G - e_0, w)$, but not a circuit chain (since $C_i \triangle D$ contains both z' and z'', and therefore, $C_{i+1} \triangle D$ is removable). $\square$

III. Let $F_t = \{f_1, \cdots, f_s\}$ be the set of all attachments of C_t in $\overline{G_{1,(t-1)}}$ and let f_0 be the attachment of e_0 in $\overline{G_{1,(t-1)}}$. Here, by Claim 6,

$$|F_t| = s \geq 2. \tag{1}$$

Notation 6.2. *(i) Construct (H, w_H) from $(G_{1,(t-1)}, w_{1,(t-1)})$ by replacing each induced path (subdivided edge f_μ) with a path of length 2. With no confusion, let each of those subdivided edges be f_μ ($\in F_t$) containing a degree 2 vertex y_μ, and x_0 is the degree 2 vertex in the subdivided edge f_0. Here, $x_0 \in f_0 \subset C_1 - C_2$ and $y_\mu \in f_\mu \subset C_{t-1} - C_{t-2}$ for each $\mu = 1, \cdots, s$.*

(ii) Construct (H_μ, w_μ) from (H, w_H) by adding a weight 2 edge e_μ joining x_0 and y_μ and suppressing all degree 2 vertices (see Fig. 15).

IV. This is the final step of this subsection.

Claim 10.

$$t = 3.$$

Proof. Suppose that $t \geq 4$.

IV-1. By applying the theorem to the smaller CCPE weighted graph (H_μ, w_μ) (for each $\mu = 1, \cdots, s$), it has the following properties:

(a) (H_μ, w_μ) is a Petersen chain with the bowstring e_μ (since any removable circuit of (H_μ, w_μ) avoiding e_μ is also removable in (G, w)).

(b) (H_μ, w_μ) does not have any 3-edge-cut T of (G, w) that consists of the bowstring e_μ and two weight one edges of C_i for some $i : 1 < i < t-1$. (Since $T - e_\mu + e_0$ would be a non-trivial 3-edge-cut of (G, w) and this contradicts Claim 5.) Thus, $E(x_0)$ and $E(y_\mu)$ are the only 3-edge-cuts of (H_μ, w_μ) containing the bowstring e_μ.

(c) (H_μ, w_μ) must be a Petersen chain with a single segment (by (b) and Lemma 5.7). Thus, $t - 1 = 3$, and so $\phi(1) = 1$ and $\phi(2) = 3$. Hence, by Lemma 5.7-(4) the segment is a P_{10}-segment.

(d) From Observation 4.4 and the discussion following Definition 4.3, (H_μ, w_μ) becomes (P_{10}, w_{10}) after a series of contractions of triangles/digons not containing x_0 or y_μ and suppressions of degree 2 vertices.

(e) Thus, in (H_μ, w_μ) the endvertex y_μ of the bowstring e_μ is not contained in any circuit of length ≤ 4, because that would result in P_{10} having a circuit of length ≤ 4 after the contractions and suppressions from (d).

IV-2. By (c), (H_μ, w_μ) is a Petersen chain with a single P_{10}-segment. To show that it must be a copy of (P_{10}, w_{10}), it suffices to show that

$$|V(H_\mu)| = 10, \tag{2}$$

for each $\mu = 1, \cdots, s$.

Suppose that $|V(H_\mu)| > 10$. By (c), (H_μ, w_μ) is a Petersen chain with single segment, but not simple (since the Petersen graph has 10 vertices). Hence, the weighted graph (H_μ, w_μ) has the following further properties:

(f) In (H_μ, w_μ), there must be some circuit(s) of length ≤ 3 (by Definition 4.3);

(g) Those triangle(s)/digon(s) described in (f) must contain some edge f_ν for $\nu \in \{1, \cdots, s\} - \{\mu\}$ (since, by Claim 4, G is of girth at least 4).

Hence, some triangle(s)/digon(s) described in (g) becomes circuit(s) of length ≤ 4 in (H_ν, w_ν) (for some $\nu \neq \mu$). This contradicts (e) in IV-1 (by a symmetric argument for replacing μ with ν) and completes the proof of Equation (2).

Thus, both (H_μ, w_μ) and (H_ν, w_ν) are copies of (P_{10}, w_{10}), which contradicts Proposition 5.5 (see Fig. 12). $\square$

6.2. *Second part of the proof: two copies of L-graphs*

By Lemma 5.9-(1), we have the following immediate corollary.

Claim 11. *For each $f_\mu \in F_t$, $(H_\mu, w_\mu) \in \langle \mathcal{K}_4 \rangle_2$.*

Claim 12. $(\overline{G_{1,2}}, w_{1,2})$ *(and $(\overline{G_{2,3}}, w_{2,3})$) is an L-graph in which the diagonal crossing chord is the attachment f_0 of e_0 as in Notation 6.2.*

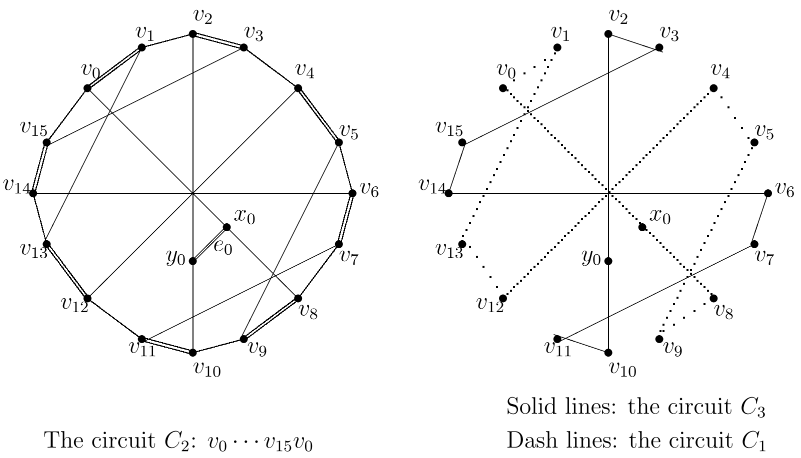

Fig. 16. $(G - e_0, w)$ is a pair of weighted L-graphs with C_2 as their overlapping part.

Proof. By Lemma 5.9-(2), we have that, *for each* $f_\mu \in F_t$*, both* f_0 *and* f_μ *are contained in some* 3*-edge-cut of* $\overline{G_{1,2}}$ (satisfying hypothesis (b) of Lemma 5.12). Since G is of girth at least 4 (by Claim 4), *every triangle/digon of* $\overline{G_{1,2}}$ *must contain some edge of* $F_t \cup \{f_0\}$ (satisfying hypothesis (a) of Lemma 5.12).

By Lemma 5.12, $(\overline{G_{1,2}}, w_{1,2})$ must be a weighted L-graph with f_0 (an attachment of e_0) as the diagonal crossing chord.

Similarly, the graph $G_{2,3} = \overline{C_2 \cup C_3}$ is also an L-graph with an attachment of e_0 as the diagonal crossing chord. $\square$

6.3. *Final step: removable circuit in* (G, w)

We continue the proof of the main theorem. The final step is the core of the proof: *determine that the graph* G *is the Petersen graph.*

6.3.1. *Preliminary*

By Claim 12, $(\overline{G_{1,2}}, w_{1,2})$ *(and* $(\overline{G_{2,3}}, w_{2,3})$*) is an* L*-graph in which the diagonal crossing chord is the attachment of* e_0.

A drawing of two L-graphs. Let $C_2 = v_0 \cdots v_{r-1}v_0$. Draw the graph $\overline{C_1 \cup C_2 \cup C_3} = \overline{G - e_0}$ on the plane such that C_2 is the boundary of the exterior region and all chords $((C_1 \cup C_3) - C_2)$ are in the interior region of C_2. (Note, this drawing is not a *planar embedding*: some crossing must occur inside the interior of C_2.)

See Fig. 16 for an illustration of these circuits in G. Note that, in the first graph of Fig. 16, double lines are edges in $E_{=2}$ and single lines are edges in $E_{w=1}$.

Notation 6.3. *(1) For each weighted* L*-graph* $\overline{C_i \cup C_2}$ *(*$i = 1, 3$*), edges of* $C_i - C_2$ *are called* C_i-chords.

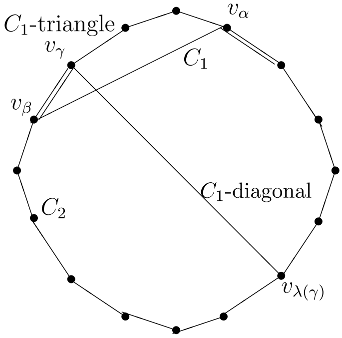

Fig. 17. A C_1-triangle $v_\alpha v_\beta v_\gamma \cdots v_\alpha$ is a circuit of length ≥ 5 in G.

(2) For each $i \in \{1,3\}$, C_i-chords are classified into two types: diagonal crossing chord and zigzag parallel chords ($Z(C_i)$-chords): the edge containing the vertex x_0 or y_0 is the C_i-diagonal crossing chord, *all other edges of $C_i - C_2$ are* $Z(C_i)$-chords *(zigzag parallel chords).*

(3) For each $\{i,j\} = \{1,3\}$, each triangle of $\overline{C_i \cup C_2}$ not containing the C_i-diagonal crossing chord is called a C_i-triangle *and the unique $Z(C_i)$-chord contained in a given C_i-triangle is called a* C_i-triangle chord *(see Fig. 17).*

By Claim 4, G is of girth at least 4, so we have the following property.

Claim 13. *For each $\{i,j\} = \{1,3\}$, let $S = v_\alpha v_\beta v_\gamma v_\alpha$ be a C_i-triangle with $v_\alpha v_\beta$ as the unique $Z(C_1)$-chord (the triangle chord), $v_\beta v_\gamma \in C_1 \cap C_2$ (see Fig. 17). Then S is a triangle in the suppressed graph $\overline{G[C_1 \cup C_2]}$, but not a triangle in the original graph G since the edge $v_\alpha v_\gamma$ of S is subdivided at least twice by vertices of C_j in G.*

Notation 6.4. *Define a mapping $\lambda : \{0, 1, \cdots, r-1\} \to \{0, 1, \cdots, r-1\}$ such that, for each integer $\alpha \in \{0, \cdots, r-1\}$, $\lambda(\alpha) = \beta$ if there is a C_i-chord (for some $i \in \{1,3\}$) joining v_α and v_β.*

Notation 6.5. *For the circuit $C_2 = v_0 \cdots v_{2k-1} v_0$, and integers a, b: $0 \leq a < b \leq 2k-1$, the segment (subpath) $v_a v_{a+1} \cdots v_{b-1} v_b$ of C_2 between v_a and v_b is denoted by $v_a C_2 v_b$, while the segment $v_a v_{a-1} \cdots v_{b+1} v_b$ of C_2 between v_a and v_b is denoted by $v_a \overline{C_2} v_b$ (mod r).*

Notation 6.6. *For each $\{i,j\} = \{1,3\}$ and each $Z(C_i)$-chord $e = v_\mu v_{\lambda(\mu)} \in C_i - C_2$ (a zigzag parallel chord belonging to C_i), the* crossing degree $d_X(e)$ *of e is the number of C_j-chords crossing the edge e in the interior of C_2.*

Since C_j is a circuit (for $\{i,j\} = \{1,3\}$), it is easy to see that

$$d_X(e) \equiv 0 \pmod 2 \tag{3}$$

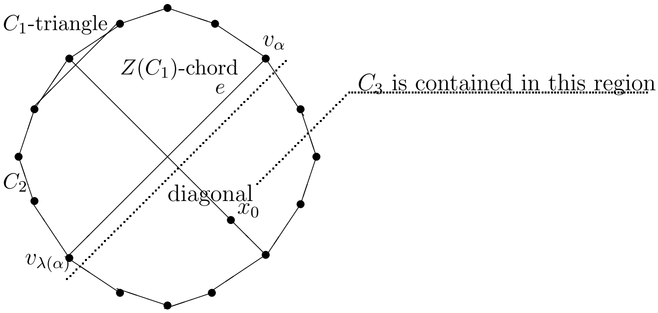

Fig. 18. If $d_X(e) = 0$ then a C_1-triangle is of length 3 in G.

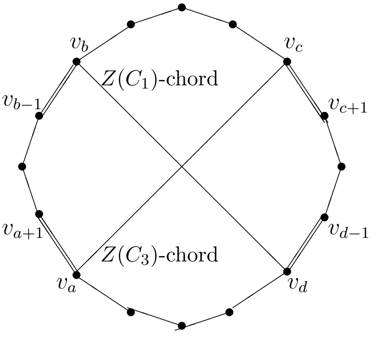

Fig. 19. A quadruple.

for every C_i-chord e (see Fig. 18). We further claim that,

$$d_X(e) > 0 \tag{4}$$

for every $Z(C_i)$-chord e.

Suppose that $d_X(e) = 0$, for some $Z(C_1)$-chord $e = v_\alpha v_{\lambda(\alpha)}$. Without loss of generality, let $V(C_3) \subseteq \{v_{\alpha+1}, v_{\alpha+2}, \cdots, v_{\lambda(\alpha)-2}, v_{\lambda(\alpha)-1}\}$. Hence, $\{v_{\lambda(\alpha)}, v_{\lambda(\alpha)+1}, \cdots, v_{\alpha-1}, v_\alpha\} \subseteq V(C_1)$. Therefore, the induced subgraph $G[\{v_{\lambda(\alpha)} \cdots v_\alpha\}]$ contains a C_1-triangle, which is not subdivided by C_3. This contradicts Claim 13.

6.3.2. Quadruples and removable circuit

Definition 6.7. Let (a, b, c, d) be a quadruple (see Fig. 19) such that

(1) v_a, v_b, v_c, v_d are around the circuit C_2 in this order;
(2) $v_a v_{a+1}$, $v_c v_{c+1} \in C_3 \cap C_2$, and $v_b v_{b-1}$, $v_d v_{d-1} \in C_1 \cap C_2$;
(3) $v_a v_c$ is a $Z(C_1)$-chord and $v_b v_d$ is a $Z(C_3)$-chord.

The proof will be completed after the proofs of the following two claims.

Claim 14. *If $(G, w) \neq (P_{10}, w_{10})$, then a quadruple described in Definition 6.7 exists.*

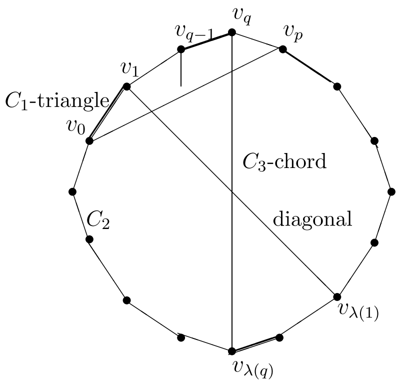

Fig. 20. C_3-chord $v_q v_{\lambda(q)}$ with one end v_q inside a C_1-triangle $v_0 v_1 v_p v_0$.

Claim 15. *If a quadruple described in Definition 6.7 exists, then the circuit* $D = v_a C_2 v_b v_d \overline{C_2} v_c v_a$ *is a removable circuit of* (G, w).

6.3.3. Existence of a quadruple (proof of Claim 14)

In this subsection, we will prove one of the following statements must be true.

(1) the existence of the quadruple described in Definition 6.7;

(2) $(G, w) = (P_{10}, w_{10})$.

Suppose that $(G, w) \neq (P_{10}, w_{10})$ and *there is no such quadruple around the circuit* C_2.

I. *Let* $v_0 v_p$ *be a* C_i*-triangle chord (*$i = 1$ *or* 3*) such that the crossing degree* d_X *of* $v_0 v_p$ *is as* **large** *as possible (among all* C_i*-triangle chords for both* $i = 1, 3$*).* Note, C_i-triangle chords are defined in Notation 6.3-(3).

Without loss of generality, let $v_0 v_1 C_2 v_p v_0$ be a C_1-triangle (see Notation 6.3) with v_1 incident with the C_1-diagonal crossing chord (see Fig. 20).

Since $v_0 v_p$ is a C_1-triangle chord, the path $v_2 C_2 v_{p-1}$ contains no vertex of C_1. By Claim 4, $v_1 C_2 v_p$ is not a single edge in G, which must contain some vertices of C_3. Therefore, by the definition of L-graph, edges in the path $v_2 C_2 v_{p-1}$ are alternatively in $C_2 - C_3$ and $C_2 \cap C_3$. That is,

$$v_2 v_3, \ \cdots, \ v_{2i} v_{2i+1}, \ \cdots, \ v_{p-2} v_{p-1} \ \in \ C_3 \cap C_2 \tag{5}$$

for $i = 1, \cdots, \frac{p-2}{2}$ and

$$p \geq 4 \text{ and } p \equiv 0 \pmod 2.$$

II.

Claim 16. *For each odd integer* $q \in \{2, \cdots, p-1\}$*, the* C_3*-chord* $v_q v_{\lambda(q)}$ *is the* C_3*-diagonal crossing chord (see Fig. 20).*

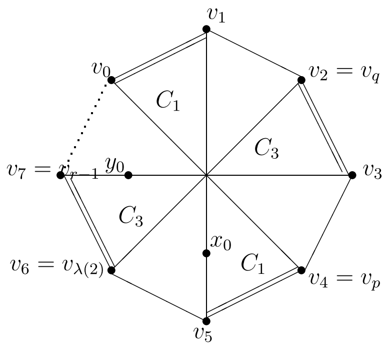

Fig. 21. C_3-chords $v_q v_{\lambda(q)}$, $v_{q+1} v_{\lambda(q+1)}$ crossing the C_1-triangle chord $v_0 v_p$.

Suppose not, then $v_q v_{\lambda(q)}$ is a $Z(C_3)$-chord. If $\lambda(q) \in \{2, \cdots, p-1\}$, then the C_3-chord $v_q v_{\lambda(q)}$ is of zero crossing degree. This contradicts Inequality (4). So, $\lambda(q) \in \{p+1, \cdots, r-1\}$. Then $(0, q, p, \lambda(q))$ is the quadruple that we needed and contradicts our assumption (see Fig. 20).

III. Since there is only one C_3-diagonal chord, by Claim 16, q is the only odd integer in $\{2, \cdots, p-1\}$. By Equation (5),

$$3 = q = p - 1.$$

IV. In summary, we have proved the following results.

(IV-1) $d_X(v_0 v_p) = 2$ *(by III);*

(IV-2) $|\{v_2, \cdots, v_{p-1}\}| = 2$ *(by III);*

(IV-3) both $v_2 v_{\lambda(2)}, v_3 v_{\lambda(3)}$ *are* C_3*-chords crossing the* $Z(C_1)$*-chord* $v_0 v_p$ *(by Claim 13);*

(IV-4) $v_3 v_{\lambda(3)}$ *is the* C_3*-diagonal crossing chord, and,* $v_2 v_{\lambda(2)}$ *is a* $Z(C_3)$*-chord (by III);*

(IV-5) $v_2 v_{\lambda(2)}$ *is a* C_3*-triangle chord (corollary of (IV-4)).*

V. By Equation (3) and Inequality (4), the crossing degree of every C_i-triangle chord is positive and even ($i = 1, 3$). By IV and the maximality of the crossing degree of the triangle chord $v_0 v_p$ (defined in I), the crossing degree of every C_i-triangle chord is precisely 2 (for each $i = 1, 3$). Hence, *all results we have had in IV for* $v_0 v_p$ *can be applied to each* C_i*-triangle chord* ($i = 1, 3$).

First, here are some direct results from IV (see Fig. 21):

$$p = 4 \quad \text{and} \quad \lambda(3) > \lambda(2) > p + 1 = 5,$$

and $v_2 v_{\lambda(2)}$ *is a* C_3*-triangle chord (by (IV-5)), and* $v_3 v_{\lambda(3)}$ *is a* C_3*-diagonal crossing chord* (by (IV-4)).

Symmetrically (see Fig. 21) we may apply the results of IV to the C_3-triangle $v_2 v_3 C_2 v_{\lambda(2)} v_2$, where $\lambda(2) = 6$ since $|\{v_4, \cdots, v_{\lambda(2)-1}\}| = 2$ (by IV-(2)). Furthermore, we

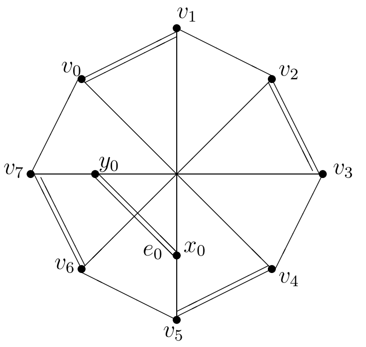

Fig. 22. The Petersen graph.

have that $v_4v_5 \in C_1 \cap C_2$, $v_0v_4v_5C_2v_0$ is a C_1-triangle (other than $v_0v_1C_2v_4v_0$) with v_0v_4 as the C_1-triangle chord, $v_5v_{\lambda(5)}$ is the C_1-diagonal crossing chord with $\lambda(5) = 1$ since there is only one C_1-diagonal chord $v_1v_{\lambda(1)} = v_{\lambda(5)}v_5$.

Note that, we have completely identified all edges of $C_1 = v_0v_1x_0v_5v_4v_0$. That is, $\overline{C_1 \cup C_2} = K_4$.

Furthermore, applying the results of IV to the C_1-triangle $v_4v_5C_2v_0v_4$, we have that $|\{v_6, \cdots, v_{r-1}\}| = 2$ and v_6v_2 is a C_3-triangle chord, $v_7v_{\lambda(7)} = v_3v_{\lambda(3)}$ is the C_3-diagonal crossing chord. Therefore, $\overline{C_3 \cup C_2} = K_4$, $r = 8$ and the graph G is the Petersen graph (see Fig. 22). This contradicts the assumption that $(G, w) \neq (P_{10}, w_{10})$.

6.3.4. Existence of removable circuit (Claim 15)

In this subsection, we will prove Claim 15 that

$$D = v_av_{a+1}C_2v_{b-1}v_bv_dv_{d-1}\overline{C_2}v_{c+1}v_cv_a$$

is a removable circuit. (See Fig. 19.) We may consider the following weight decomposition (Definition 2.4)

$$(G, w) = (G_1, w_1) + (D, w_D)$$

where $w_D(e) = 1$ if $e \in E(D)$. Since D is a circuit, it is trivial that w_1 is a $(1, 2)$-eulerian weight of G_1. So, we only need to show that G_1 is bridgeless. *Assume that there is a bridge e^* of G_1 with $w_1(e^*) = 2$* (since w_1 is eulerian).

I. Let $G_1' = G_1 - e_0$. It is obvious that G_1' is covered by paths:

$$P_0 = v_{b-1}v_bC_2v_cv_{c+1},\ P_2 = v_{d-1}v_dC_2v_av_{a+1},$$
$$P_1 = C_1 - \{v_b, v_d\} = v_{d-1}C_1v_{b-1},\ P_3 = C_3 - \{v_a, v_c\} = v_{a+1}C_3v_{c+1}.$$

(See Fig. 19.) Note that P_0 and P_2 are two segments (subpaths) of C_2 (by deleting some edges of D), and P_i is a segment of C_i for $i = 1$ and 3. It is easy to see that

$v_{c+1}P_0v_{b-1}P_1v_{d-1}P_2v_{a+1}P_3v_{c+1}$ is a closed walk of G'_1 covering every edge e once if $w_1(e) = 1$, twice if $w_1(e) = 2$.

By the discussion above and the structure of weighted L-graphs $(\overline{C_1 \cup C_2}, w_{1,2})$, $(\overline{C_2 \cup C_3}, w_{1,2})$, we have the following summary:

(I-1) *G'_1 is a connected graph, so is G_1;*

(I-2) *Paths of $\{P_0, \cdots, P_3\}$ have the following set of the endvertices*

$$\{v_{a+1}, v_{b-1}, v_{c+1}, v_{d-1}\}$$

where

$$v_{b-1} \in P_0 \cap P_1,\ v_{d-1} \in P_1 \cap P_2,\ v_{a+1} \in P_2 \cap P_3,\ v_{c+1} \in P_3 \cap P_0;$$

(I-3) *$P_i \cap P_j = \emptyset$ if $i \neq j \pm 1$ (mod 4).*

II. Let R_1, R_2 be components of $G_1 - e^*$.

(II-1) *The edge $e_0 = x_0y_0$ is not a bridge of G_1* (that is, $e^* \neq e_0$) since $G'_1 = G_1 - e_0$ is connected (by (I-1)). Thus, $e^* \neq e_0$.

(II-2) Since $w_1(e^*) = 2$, by (II-1), let P_α, P_β ($\in \{P_0, \cdots, P_3\}$) contain the edge e^*. By (I-3), we have that $\alpha = \beta \pm 1$ (mod 4). Without loss of generality, let $e^* \in P_0 \cap P_1$. It is easy to see that P_2 and P_3 must be contained in the same component of $G_1 - e^*$ since $v_{a+1} \in V(P_2) \cap V(P_3)$ (by (I-2)). So, without loss of generality, let $P_2 \cup P_3 \subseteq R_2$. Therefore, by (I-2) again,

$$v_{c+1}, v_{d-1}, v_{a+1} \in R_2.$$

Since each of P_0 and P_1 passes through the bridge e^* precisely once,

$$v_{b-1} \in R_1.$$

(II-3) Let $e^* = v_qv_{q+1}$ where

$$v_q, v_{q+1} \in \{v_{b+1}, v_{b+2}, \cdots, v_{c-2}, v_{c-1}\} \subseteq P_0 \subset C_2.$$

For the path $P_0 = v_{b-1}C_2v_{c+1}$, by (II-2), the segments $v_{b-1}C_2v_q \subseteq R_1$ and $v_{q+1}C_2v_{c+1} \subseteq R_2$.

Note that the segment $v_{b-1}C_2v_q$ is contained in R_1 while C_3 is contained in R_2. Thus, $v_{b-1}C_2v_q$ contains vertices of C_1, but not C_3.

(II-4) We claim that $\underline{v_{d-1}v_{\lambda(d-1)}}$ is not the C_1-diagonal (see Fig. 23,) for otherwise, $v_{d-1}v_dv_bC_2v_{d-1}$ is a C_1-triangle of $\overline{C_1 \cup C_2}$ and therefore, the segment $v_bC_2v_{d-1}$ contains no edges of $C_1 \cap C_2$. This contradicts that v_qv_{q+1} ($\in P_1 \cap P_2 \subset C_1 \cap C_2$) lies in the segment of C_2 from v_b to v_{d-1}.

Since, both v_dv_b and $v_{d-1}v_{\lambda(d-1)}$ are $Z(C_1)$-chords and the vertex $v_{\lambda(d-1)}$ must be in $\{v_b, v_{b+1}, \cdots, v_q\}$. That is, according to the structure of L-graph, $\lambda(d-1) = b+1 \leq q$.

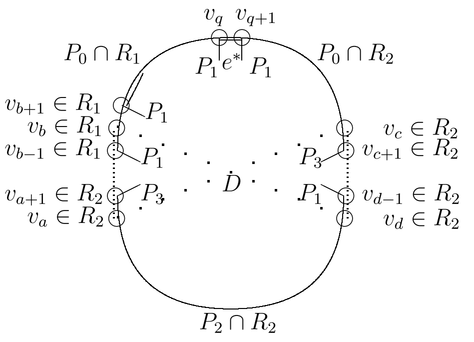

Fig. 23. The $Z(C_1)$-chord $v_{d-1}v_{\lambda(d-1)} = v_{d-1}v_{b+1}$.

Furthermore, this edge $v_{d-1}v_{\lambda(d-1)} = v_{d-1}v_{b+1}$ joins the components R_1 and R_2 since $v_{d-1} \in R_2$ while $v_{\lambda(d-1)} = v_{b+1} \in R_1$. This contradicts that $e^* = v_q v_{q+1}$ is a bridge of $G - E(D)$. This completes the proof of Claim 15, and also completes the proof of the theorem: we have obtained a contradiction to the assumption that (G, w) has no removable cycle not containing e_0.

References

[1] B. Alspach, C.-Q. Zhang, Cycle covers of cubic multigraphs, Discrete Math. 111 (1993) 11–17.
[2] B. Alspach, L.A. Goddyn, C.-Q. Zhang, Graphs with the circuit cover property, Trans. Amer. Math. Soc. 344 (1) (1994) 131–154.
[3] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1976.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.
[5] R. Diestel, Graph Theory, 3rd ed., Springer-Verlag, Heidelberg, ISBN 3-540-26183-4, 2000.
[6] M.N. Ellingham, Petersen subdivisions in some regular graphs, Congr. Numer. 44 (Nov. 1984) 33–40.
[7] H. Fleischner, Some blood, sweat, but no tears in eulerian graph theory, Congr. Numer. 63 (Nov. 1988) 9–48.
[8] H. Fleischner, F. Genest, B. Jackson, Compatible circuit decompositions of 4-regular graphs, J. Graph Theory 56 (2007) 227–240.
[9] L.A. Goddyn, Cycle double covers-current status and new approaches, in: Contributed Lecture at Cycle Double Cover Conjecture Workshop, IINFORM, Vienna, 1991.
[10] L.A. Goddyn, J. van den Heuvel, S. McGuinness, Removable circuits in multigraphs, J. Combin. Theory Ser. B 71 (1997) 130–143.
[11] A. Itai, M. Rodeh, Covering a graph by circuits, in: Automata, Languages and Programming, in: Lecture Notes in Computer Science, vol. 62, Springer-Verlag, Berlin, 1978, pp. 289–299.
[12] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, in: Keele (Ed.), Surveys in Combinatorics, in: London Math. Soc. Lecture Note Ser., vol. 187, Cambridge Univ. Press, Cambridge, 1993, pp. 191–210.
[13] F. Jaeger, A survey of the cycle double cover conjecture, in: B. Alspach, C. Godsil (Eds.), Cycles in Graphs, in: Ann. Discrete Math., vol. 27, 1985, pp. 1–12.
[14] H.-J. Lai, C.-Q. Zhang, Hamilton weight and Petersen minor, J. Graph Theory 38 (2001) 197–219.
[15] N. Robertson, P.D. Seymour, R. Thomas, Tutte's edge-colouring conjecture, J. Combin. Theory Ser. B 70 (1) (1997) 166–183.
[16] P.D. Seymour, Sums of circuits, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, pp. 342–355.
[17] P.D. Seymour, K. Truemper, A Petersen on a pentagon, J. Combin. Theory Ser. B 72 (1) (1998) 63–79.
[18] G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Aust. Math. Soc. 8 (1973) 367–387.
[19] W.T. Tutte, Personal correspondence with H. Fleischner, July 22, 1987.

[20] D. West, Introduction to Graph Theory, 2nd ed., Prentice Hall, ISBN 0-13-014400-2, 2000.
[21] C.-Q. Zhang, Hamiltonian weights and unique 3-edge-colorings of cubic graphs, J. Graph Theory 20 (1) (1995) 91–99.
[22] C.-Q. Zhang, Integer Flows and Cycle Covers of Graphs, Marcel Dekker, 1997.
[23] C.-Q. Zhang, Cycle covers (I) – minimal contra pairs and Hamilton weights, J. Combin. Theory Ser. B 100 (2010) 419–438.