Journal of Combinatorial Theory, Series B 103 (2013) 587–598

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Nowhere-zero 3-flows and modulo k-orientations ☆

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ARTICLE INFO

Article history:
Received 30 October 2011
Available online 20 August 2013

Keywords:
Integer flow
3-flow
Tutte orientation
Modulo 3-orientation

ABSTRACT

The main theorem of this paper provides partial results on some major open problems in graph theory, such as Tutte's 3-flow conjecture (from the 1970s) that every 4-edge connected graph admits a nowhere-zero 3-flow, the conjecture of Jaeger, Linial, Payan and Tarsi (1992) that every 5-edge-connected graph is Z_3-connected, Jaeger's circular flow conjecture (1984) that for every odd natural number $k \geqslant 3$, every $(2k-2)$-edge-connected graph has a modulo k-orientation, etc. It was proved recently by Thomassen that, for every odd number $k \geqslant 3$, every $(2k^2+k)$-edge-connected graph G has a modulo k-orientation; and every 8-edge-connected graph G is Z_3-connected and admits therefore a nowhere-zero 3-flow. In the present paper, Thomassen's method is refined to prove the following: *For every odd number $k \geqslant 3$, every $(3k-3)$-edge-connected graph has a modulo k-orientation*. As a special case of the main result, *every 6-edge-connected graph is Z_3-connected and admits therefore a nowhere-zero 3-flow*. Note that it was proved by Kochol (2001) that it suffices to prove the 3-flow conjecture for 5-edge-connected graphs.

1. Introduction

All graphs considered in this paper are allowed to have multiple edges.

☆ This research project is partially supported by an NSA grant H98230-12-1-0233 and an NSF grant DMS-126480.

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0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2013.06.003

1.1. The 3-flow conjecture

Integer flow was originally introduced by Tutte [60,61] as a generalization of map coloring.

Definition 1.1. Let G be a graph and k be a natural number. A pair (D, f) is an *integer k-flow* of G if D is an orientation of G and $f : E(G) \mapsto \{0, \pm 1, \ldots, \pm(k-1)\}$ such that

$$\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$$

for every vertex $v \in V(G)$. Furthermore, a k-flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$.

One major open problem in the integer flow theory is the following conjecture, which is the dual version of Grötzsch's 3-color theorem for planar graphs (see [1,11,18,19,31,56,63]).

Conjecture 1.2 *(Tutte). Every* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Conjecture 1.2 is called *Tutte's* 3*-flow conjecture* and appeared in the 1970s, see e.g. [52] and [6] (Open Problem 48).

Kochol [33] proved that *it suffices to prove the* 3*-flow conjecture for* 5*-edge-connected graphs*.

A weakened version of Conjecture 1.2, the so-called *weak* 3*-flow conjecture*, was proposed by Jaeger.

Conjecture 1.3. *(Jaeger [27].) There is a natural number h such that every h-edge-connected graph admits a nowhere-zero* 3*-flow.*

Theorems 1.4 and 1.5 below are early partial results on Conjectures 1.2 and 1.3.

Theorem 1.4. *(Lai and Zhang [39].) Every $4\lceil \log_2 n_o \rceil$-edge-connected graph with at most n_o odd-degree vertices admits a nowhere-zero 3-flow.*

Theorem 1.5. *(Alon, Linial and Meshulam [3], see also [2].) Every $2\lceil \log_2 n \rceil$-edge-connected graph with n vertices admits a nowhere-zero* 3*-flow.*

Conjecture 1.3 was recently verified by Thomassen.

Theorem 1.6. *(Thomassen [57].) Every* 8*-edge-connected graph admits a nowhere-zero* 3*-flow.*

This theorem is further improved in this paper as follows.

Theorem 1.7. *Every* 6*-edge-connected graph admits a nowhere-zero* 3*-flow.*

In Section 4.4, Theorem 1.7 is further strengthened by replacing edge-connectivity by odd-edge-connectivity.

1.2. Modulo orientation

Definition 1.8. Let G be a graph, let k be an odd integer ($k \geqslant 3$) and let Z_k be the cyclic group of order k. An orientation D of G is called a *modulo k-orientation* if, for every vertex $v \in V(G)$,

$$d^+_D(v) \equiv d^-_D(v) \pmod{k}.$$

Definition 1.9. Let G be a graph, and let k be an odd integer ($k \geqslant 3$).

(i) A mapping $\beta : V(G) \mapsto Z_k$ is called a *Z_k-boundary* of G if $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{k}$.

(ii) Let β be a Z_k-boundary of G. An orientation D of G is called a *β-orientation* if, for every vertex $v \in V(G)$,

$$d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}.$$

See Sections 4.2 and 4.3 for further discussion about the relations between modulo orientations and flow problems.

Conjecture 1.10. *Let G be a graph, and let k be an odd integer ($k \geqslant 3$).*

(i) (*Jaeger* [28], *also see* [29,68]) *If G is $(2k-2)$-edge-connected, then G has a modulo k-orientation*;

(ii) (*Seymour* [49], *and, Galluccio, Goddyn, Hell* [16], *see also* [69]) *There exists a natural number $f(k)$ such that every $f(k)$-edge-connected graph has a modulo k-orientation*;

(iii) (*Jaeger, Linial, Payan and Tarsi* [30]) *If G is 5-edge-connected, then G has a β-orientation for every Z_3-boundary β of G*;

(iv) (*Lai* [38], *see also* [43]) *If G is $(2k-1)$-edge-connected, then G has a β-orientation for every Z_k-boundary β of G*;

(v) (*Lai* [38], *see also* [43]) *There exists a natural number $g(k)$ such that every $g(k)$-edge-connected graph has a β-orientation for every Z_k-boundary β.*

Conjecture 1.10(ii) and (v) have been proved recently by Thomassen [57].

Theorem 1.11. *(Thomassen [57].) Let k be an odd integer ($k \geqslant 3$). Every $(2k^2+k)$-edge-connected graph G has a β-orientation for every Z_k-boundary β of G.*

In this paper, the quadratic function for the edge-connectivity in Theorem 1.11 is reduced to a linear function.

Theorem 1.12. *Let k be an odd integer $\geqslant 3$. Every $(3k-3)$-edge-connected graph G has a β-orientation for every Z_k-boundary β of G.*

The sharpness of the result will be discussed in Section 5.1.

There is a long list of publications on 3-flow, modulo k-orientation with or without boundary, such as, [1–4,8–10,12–14,16,18,19,21–30,32–39,41–48,50,52–56,59,60,65–67,69–72], etc. Many of those papers are for graphs with special properties (instead of edge-connectivity) such as local density, local structure, random structure, symmetry properties, embedding properties, degree conditions, and odd-cuts distribution. Some of these results remain the best known results for the graph families they concern, and are not corollaries of the results in the present paper.

2. Preliminaries

2.1. The set mapping $\tau(A)$

The idea which makes the proof in [57] work is a set function called $t(A)$ with values in $\{0, 1, 2, \ldots, k\}$, where k is a natural number, $k \geqslant 3$. In the present paper we assume that k is odd, and we use a similar mapping that we allow to have negative values. We therefore call it $\tau(A)$. This mapping has values in $\{0, \pm1, \pm2, \ldots, \pm k\}$, and $t(A) = |\tau(A)|$.

Suppose $\beta : V(G) \mapsto Z_k = \{0, 1, 2, \ldots, k-1\}$ such that $\sum_{v\in V(G)} \beta(v) \equiv 0 \pmod{k}$. As in Definition 1.9, we call β a boundary of the graph G. Define the mapping $\tau : V(G) \mapsto \{0, \pm1, \pm2, \ldots, \pm k\}$ such that, for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} \beta(x) & \pmod{k}, \\ d(x) & \pmod{2}. \end{cases} \tag{1}$$

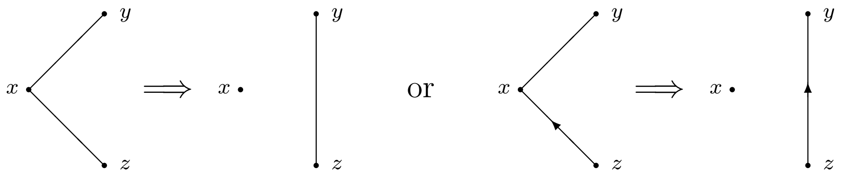

Fig. 1. Lifting of xy and xz.

Let x be a vertex of G of degree at least $|\tau(x)|$. Then $d(x) - |\tau(x)|$ is even, and there is a natural way to direct the edges incident with x (we call this set of edges $E(x)$) such that $d^+(x) - d^-(x) \equiv \beta(x) \pmod k$: First we choose $\frac{d(x)-|\tau(x)|}{2}$ pairs of edges and direct each pair in opposite directions. Then we direct all the remaining $|\tau(x)|$ edges away from x (if $\tau(x) \geqslant 0$) or all toward x (if $\tau(x) \leqslant 0$). Such an orientation of $E(x)$ satisfies the relation $d^+(x) - d^-(x) \equiv \beta(x) \pmod k$ since $d^+(x) = \frac{d(x)+\tau(x)}{2}$ and $d^-(x) = \frac{d(x)-\tau(x)}{2}$.

Notice that, if $\beta(x) = 0$ and $d(x)$ is odd, then $|\tau(x)| = k$, and we can direct $\frac{d(x)-k}{2}$ pairs of edges in opposite directions and the remaining k edges either all away from x or all toward x. So in this case we may put $\tau(x) = -k$, and we may also put $\tau(x) = k$. Moreover,

$$\tau(x) = \begin{cases} \beta(x) & \text{if } d(x) - \beta(x) \text{ is even;} \\ \beta(x) - k & \text{if } d(x) - \beta(x) \text{ is odd.} \end{cases} \tag{2}$$

The mapping τ extends to any nonempty vertex subset A with respect to $\beta(A) \equiv \sum_{x \in A} \beta(x) \pmod k \in \{0, 1, \ldots, k-1\}$ and $d(A) = |[A, V(G) \backslash A]|$, where $[A, V(G) \backslash A]$ is the set of edges between A and $V(G) \backslash A$. The extended mapping $\tau : \mathcal{P}(V(G)) \mapsto \{0, \pm 1, \pm 2, \ldots, \pm k\}$ is defined as follows, for each nonempty $A \subset V(G)$:

$$\tau(A) \equiv \begin{cases} \beta(A) & \pmod k, \\ d(A) & \pmod 2, \end{cases}$$

where $\mathcal{P}(V(G))$ is the power set of $V(G)$, that is, the collection of all subsets of $V(G)$.

Proposition 2.1. *Let A be a vertex subset of the graph G.*

$$\text{If } d(A) \geqslant 3k - 3, \quad \text{then } d(A) \geqslant (2k-2) + |\tau(A)|.$$

Proposition 2.1 follows from the fact that $|\tau(A)| \leqslant k$ and $d(A) - |\tau(A)|$ is even.

2.2. Liftings

Lifting. Let x be a vertex of G. If xy and xz are two edges, and y and z are distinct vertices, then the deletion of the edges xy and xz and the addition of the edge yz is called the *lifting* of xy and xz (see Fig. 1). Also, if one of xy and xz, say xz, is a directed edge, then we direct yz toward z if xz is toward z or away from z otherwise.

Observation. Let G' be the graph constructed from G by lifting two edges xy and xz. Then every orientation of G' can be considered as an orientation of G with xy and xz oriented in opposite directions from x.

3. The main result

All results in this paper are corollaries of the following technical result, which is a refinement of Theorem 2 in [57]. An additional new idea, due to Wu [64], is an additional vertex v_0 which we allow to have small degree.

Theorem 3.1. *Let G be a graph with Z_k-boundary β. Let z_0 be a vertex of $V(G)$, and let D_{z_0} be a pre-orientation of $E(z_0)$. Let $V_0 = \{v \in V(G) - z_0 \mid \tau(v) = 0\}$. If $V_0 \neq \emptyset$, we let v_0 be a vertex of V_0 with smallest degree. Assume that*

(i) $|V(G)| \geqslant 3$.

(ii) $d(z_0) \leqslant (2k-2) + |\tau(z_0)|$, *and the edges incident with z_0 are pre-directed such that* $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod k$.

(iii) $d(A) \geqslant (2k-2) + |\tau(A)|$ *for each nonempty vertex subset A which does not contain z_0 and which satisfies the conditions that $A \neq \{v_0\}$ and $|V(G)\backslash A| > 1$.*

Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to a β-orientation D of the entire graph G, that is, for each vertex x of $V(G)$,

$$d^+(x) - d^-(x) \equiv \beta(x) \pmod k.$$

Theorem 1.12 is a corollary of Theorem 3.1 since (by Proposition 2.1) a $(3k-3)$-edge-connected graph with any boundary satisfies the conditions of Theorem 3.1 after adding an additional vertex z_0. Moreover, Theorem 1.7 is a corollary of Theorem 1.12 since a graph admits a nowhere-zero 3-flow if and only if it has a modulo 3-orientation.

3.1. Proof of Theorem 3.1

The proof is by contradiction. We assume (reductio ad absurdum) that (G, β, z_0) is a counterexample. That is, the graph G with Z_k-boundary β satisfies the conditions of the theorem but some pre-orientation D_{z_0} cannot be extended to a β-orientation of G.

Let $\mathcal{M}$ be the collection of counterexamples (G, β, z_0) such that $|V(G)| + |E(G - z_0)|$ is minimum.

The proof is divided into two parts. The first part, Claims 1–5 below, establishes some properties of all members of $\mathcal{M}$. In the second part we choose a member (G, β, z_0) of $\mathcal{M}$ such that $|E(G)|$ is minimum and prove that it is not a counterexample, yielding a contradiction.

If we work with distinct graphs G, G', we use the terms $d(A)$, $\beta(A)$ and $\tau(A)$ when A is a vertex subset of G, and $d'(A)$, $\beta'(A)$ and $\tau'(A)$ when A is a vertex subset of G'.

Part I. Some properties of $\mathcal{M}$.

In Part 1 we let (G, β, z_0) be any member of $\mathcal{M}$.

Claim 1. *If $A \subseteq V(G) - z_0$ is a vertex subset such that $1 < |A| < |V(G) - z_0|$, then*

$$d(A) \geqslant 2k + \left|\tau(A)\right|.$$

If $d(A) < 2k + |\tau(A)|$, then we first get an extension of D_{z_0} to the contracted graph $H = G/A$ by the minimality property of G, since $|V(H)| < |V(G)|$ and $|E(H - z_0)| \leqslant |E(G - z_0)|$. Then all edges of the edge-cut $[A, A^c]$, where $A^c = V(G)\backslash A$, are oriented in this extension. Similarly we then contract A^c into a single vertex as a new z_0, and again, we use the minimality of G to extend the orientation of $[A, A^c]$ to the edges in $G[A]$. $\square$

Claim 2. $V_0 = \emptyset$.

Suppose $V_0 \neq \emptyset$ and v_0 is a vertex of V_0 with smallest degree.

We can assume that $d(v_0) \geqslant 2$. Otherwise v_0 is an isolated vertex.

If v_0 has at least two neighbors, we lift one pair of edges incident with v_0. Claim 1 implies that the resulting graph G' satisfies the hypotheses of the theorem. Since $|E(G' - z_0)| < |E(G - z_0)|$, it holds that G' has the desired orientation, and by the Observation in Section 2.2, so does G, a contradiction.

Now suppose v_0 has only one neighbor x.

We must have $x \neq z_0$. Otherwise, $\tau(W) = -\tau(\{z_0, v_0\}) = -\tau(z_0)$, where $W = V(G) - z_0 - v_0$, and then $d(z_0) = d(W) + d(v_0) \geqslant (2k-2) + |\tau(W)| + 2 = 2k + |\tau(z_0)|$, a contradiction to condition (ii).

Let $G' = G - v_0$. For each nonempty vertex subset A of G' not containing z_0 such that $|V(G')\backslash A| > 1$, we have $d'(A) = d(A)$ or $d(A + v_0)$. Since $\tau'(A) = \tau(A) = \tau(A + v_0)$, it follows that A satisfies condition (iii).

If $|V(G)| > 3$, then $|V(G')| \geqslant 3$, and the minimality of G implies that G' has an orientation D' satisfying the theorem. If $|V(G)| = 3$, we let $D' = D_{z_0}$. Then we extend D' to an orientation of G by orienting half of the edges between x and v_0 toward v_0 and the other half away from v_0, yielding a contradiction. □

Claim 3. *$G - z_0$ is connected, and $d(z_0) \geqslant k$.*

Suppose $G - z_0$ is disconnected and let U and W be two components of $G - z_0$. By condition (iii) and Claim 2, we have $d(U) \geqslant 2k - 2$ and $d(W) \geqslant 2k - 2$. Then

$$d(z_0) \geqslant d(U) + d(W) > (2k - 2) + k \geqslant (2k - 2) + \left|\tau(z_0)\right|,$$

a contradiction to condition (ii).

Suppose $d(z_0) \leqslant k - 1$ and let G' be the graph constructed from G by replacing an edge xy of $G - z_0$ with a directed path of length two through z_0 with $\beta' = \beta$. We have $d'(z_0) \leqslant (k - 1) + 2 \leqslant 2k - 2 \leqslant (2k - 2) + |\tau'(z_0)|$ and hence G' satisfies condition (ii). For any vertex subset A described in condition (iii), $d'(A) = d(A) + 2$ if A contains both x and y, and $d'(A) = d(A)$ otherwise. So condition (iii) is clearly satisfied. Since $|V(G')| = |V(G)|$ and $|E(G' - z_0)| < |E(G - z_0)|$, this implies (by the definition of $\mathcal{M}$) that an extension of D_{z_0} exists in G'. This orientation results in an orientation of G, a contradiction. □

Claim 4. *For any two distinct vertices $x, y \in V(G) - z_0$, we have $\tau(x)\tau(y) > 0$.*

Suppose $\tau(x)\tau(y) \leqslant 0$. By Claim 2, we may assume that $\tau(x) > 0$ and $\tau(y) < 0$.

By Claim 3, $G - z_0$ has a path joining x and y. So there exists an edge x_1x_2 of the path such that $\tau(x_1) > 0$ and $\tau(x_2) < 0$. We delete x_1x_2, decrease $\beta(x_1)$ by 1 and increase $\beta(x_2)$ by 1. Let $G' = G - x_1x_2$, and let β' be the modified boundary.

Then $|V(G')| = |V(G)|$ and $|E(G' - z_0)| < |E(G - z_0)|$. If G' and β' satisfy the conditions of the theorem, then by the definition of $\mathcal{M}$, the pre-orientation can be extended to a β'-orientation of G' and further to a β-orientation of G by adding a directed edge from x_1 to x_2, yielding a contradiction. Hence, it suffices to verify the conditions of the theorem for G' and β'. Moreover, we only need to verify condition (iii) for single vertices x_1 and x_2 and vertex subsets A such that $|A| > 1$ and $d'(A) = d(A) - 1$ which are affected by the deletion of x_1x_2.

Condition (iii) is satisfied for x_i since, for $i = 1, 2$, $d'(x_i) = d(x_i) - 1$, $\beta'(x_1) = \beta(x_1) - 1$, $\beta'(x_2) = \beta(x_2) + 1$, $0 \leqslant \tau'(x_1) = \tau(x_1) - 1$, $0 \geqslant \tau'(x_2) = \tau(x_2) + 1$, and therefore,

$$\left|\tau'(x_i)\right| = \left|\tau(x_i)\right| - 1.$$

The equation $|\tau'(x_i)| = |\tau(x_i)| - 1$ also holds when $\tau(x_i) = \pm k$. For any vertex subset A (in condition (iii)) such that $|A| > 1$ and $d'(A) = d(A) - 1$, we have $|\tau'(A)| = |\tau(A) \pm 1| \leqslant |\tau(A)| + 1$ and by Claim 1, $d'(A) \geqslant 2k + |\tau(A)| - 1 \geqslant (2k - 2) + |\tau'(A)|$. Hence condition (iii) is verified for A.

So $\tau(x)\tau(y) > 0$. □

Let

$$V^+ = \left\{x \in V(G) - z_0\colon\ 1 \leqslant \tau(x) \leqslant k - 1\right\}$$

and

$$V^- = \left\{x \in V(G) - z_0\colon\ 1 - k \leqslant \tau(x) \leqslant -1\right\}.$$

Note that, if $\beta(x) = 0$ and $d(x)$ is odd, then x has two possible τ-values, namely k and $-k$.

Claim 5. *$V(G) - z_0 = V^+$ or $V(G) - z_0 = V^-$.*

By Claim 4, we have $V^+ = \emptyset$ or $V^- = \emptyset$. So it suffices to prove that $|\tau(x)| \neq k$ for any vertex x other than z_0.

If $x \in V(G) - z_0$ such that $|\tau(x)| = k$, then for any vertex y distinct from x and z_0, we can choose $\tau(x) = k$ or $\tau(x) = -k$ such that $\tau(x)\tau(y) \leqslant 0$ and get a contradiction to Claim 4. $\square$

Part II. Minimum members of $\mathcal{M}$.

Now choose (G, β, z_0) to be a member of $\mathcal{M}$ such that $|E(G)|$ is minimum.

Without loss of generality, assume that $V(G) - z_0 = V^+$. For if $V(G) - z_0 = V^-$, we reverse the directions of all edges incident with z_0 and replace $\beta(x)$ by $k - \beta(x)$ for each vertex x (including z_0) with $\beta(x) \neq 0$. Then the resulting graph with the modified boundary satisfies $V(G) - z_0 = V^+$ and is also a minimum member of $\mathcal{M}$.

For each vertex $x \in V(G) - z_0$,

$$d(x) \geqslant (2k-2) + \tau(x) \quad \text{with } 1 \leqslant \tau(x) = \beta(x) \leqslant k-1. \tag{3}$$

Claim 6. *$d(z_0) = k + \beta(z_0)$, and all edges incident with z_0 are directed away from z_0.*

By Claim 3, z_0 has a neighbor x. By Claim 5, $1 \leqslant \tau(x) \leqslant k-1$.

If xz_0 is directed towards z_0, then we delete xz_0, decrease $\beta(x)$ by 1 and increase $\beta(z_0)$ by 1. By a proof similar to that of Claim 4, the resulting graph with modified boundary satisfies the conditions of the theorem. Since $|V(G')| = |V(G)|$ and $|E(G')| < |E(G)|$, and (G, β, z_0) is a smallest member of $\mathcal{M}$, the pre-orientation can be extended to a β'-orientation of G' and then to a β-orientation of G which contradicts the fact that (G, β, z_0) is a counterexample.

So all edges incident with z_0 are directed away from z_0, and $\beta(z_0) = d(z_0) - sk$, where s is the biggest integer such that $d(z_0) - sk \geqslant 0$.

By condition (ii), $d(z_0) \leqslant (2k-2) + |\tau(z_0)| \leqslant 3k-2$. If $2k \leqslant d(z_0) \leqslant 3k-2$, then $s = 2$ and $d(z_0) - \beta(z_0)$ is even. By (2), $\tau(z_0) = \beta(z_0)$ and then $d(z_0) = 2k + |\tau(z_0)|$, a contradiction to condition (ii). So $k \leqslant d(z_0) \leqslant 2k-1$, and $d(z_0) = k + \beta(z_0)$. $\square$

The final step: (G, β, z_0) is not a counterexample.

By Claim 6, let x be a neighbor of z_0, and let e be an edge directed from z_0 to x. We replace e by $k-1$ multiple directed edges from x to z_0.

Let G' be the resulting graph with $\beta' = \beta$. We are going to prove that G' with the boundary β' satisfies all conditions of the theorem and, furthermore, $1 - k \leqslant \tau'(x) \leqslant -1$ for the vertex x.

We have $\beta'(z_0) = \beta(z_0)$ and $d'(z_0) = d(z_0) + k - 2$. By Claim 6, $d(z_0) = k + \beta(z_0)$. Then $d'(z_0) = (k + \beta(z_0)) + (k-2) = (2k-2) + \beta'(z_0)$. By (2), we have $\tau'(z_0) = \beta'(z_0)$ and therefore, $d'(z_0) = (2k-2) + |\tau'(z_0)|$. So, condition (ii) is satisfied for G' and β'.

For condition (iii), we only need to consider x and vertex subsets containing x. $d'(x) = d(x) + (k-2) \geqslant (2k-2) + |\tau(x)| + (k-2) \geqslant 3k-3$ since by Claim 2, $\tau(x) \neq 0$. By Proposition 2.1, $d'(x) \geqslant (2k-2) + |\tau'(x)|$. By Claim 1, for any non-trivial vertex subset A of G' described in condition (iii) and containing x, we have $d'(A) = d(A) + k - 2 \geqslant 3k-2 = (2k-2) + k \geqslant (2k-2) + |\tau'(A)|$. So, condition (iii) is also satisfied.

By (3), $1 \leqslant \tau(x) = \beta(x) \leqslant k-1$ and by (1), $d(x) - \tau(x)$ is even. So $d'(x) - \beta'(x) = (d(x) + (k-2)) - \beta(x) = (d(x) - \tau(x)) + (k-2)$ and is odd. Then by (2), $\tau'(x) = \beta'(x) - k = \beta(x) - k$ and $1 - k \leqslant \tau'(x) \leqslant -1$.

Now if (G', β', z_0) is also a counterexample, then $(G', \beta', z_0) \in \mathcal{M}$ since $|V(G')| = |V(G)|$ and $|E(G' - z_0)| = |E(G - z_0)|$. But we have $V'^+ = V(G') - z_0 - x$ and $V'^- = \{x\}$, a contradiction to Claim 5.

So (G', β', z_0) is not a counterexample, and hence G' has a β'-orientation. Then the corresponding orientation of G (obtained by replacing the $k-1$ edges from x to z_0 with one edge in opposite direction) is a β-orientation of G satisfying the theorem.

This completes the proof. $\square$

4. Applications to graph flow, group connectivity and odd-edge-connectivity

4.1. The 5-flow conjecture

The 5-flow conjecture, a dual version of the 5-color theorem (Heawood [20]) for planar graphs, is another major open problem in graph theory.

Conjecture 4.1. *(Tutte [61].) Every bridgeless graph admits a nowhere-zero* 5*-flow.*

The best partial result on this conjecture, due to Seymour [48], says that *every bridgeless graph admits a nowhere-zero* 6*-flow*. A different approach to Conjecture 4.1 was proposed by Jaeger.

Proposition 4.2. *(Jaeger [29].) If every* 9*-edge-connected graph has a modulo* 5*-orientation, then every bridgeless graph admits a nowhere-zero* 5*-flow.*

Thus Conjecture 1.10(i) implies the 5-flow conjecture. The first weak version of Proposition 4.2 is the result of Thomassen [57] that *every* 55*-edge-connected graph has a modulo* 5*-orientation*. The result in the present paper implies that *every* 12*-edge-connected graph has a modulo* 5*-orientation*.

4.2. Circular flows

Definition 4.3. Let a, b be two integers such that $0 < b \leqslant \frac{a}{2}$. An integer flow (D, f) of a graph G is called *a circular* $\frac{a}{b}$*-flow* if $f : E(G) \mapsto \{\pm b, \pm(b+1), \dots, \pm(a-b)\} \cup \{0\}$.

The concept of circular flows, introduced in [17], is a generalization of integer flows, and a dual version of circular colorings [5,62]. We refer to [72,73] for surveys.

Proposition 4.4. *(Jaeger [28].) For every positive integer p, a graph G admits a nowhere-zero circular $(2+\frac{1}{p})$-flow if and only if G has a modulo $(2p+1)$-orientation.*

The following is an immediate corollary of Proposition 4.4 and Theorem 1.12.

Corollary 4.5. *For every natural number p, every $6p$-edge-connected graph admits a nowhere-zero circular $(2+\frac{1}{p})$-flow.*

It is proved in [17] (also see [68, Lemma 9.3.3]) that if a graph G admits a nowhere-zero circular r-flow, then G admits a nowhere-zero circular s-flow for every pair of rational numbers $s \geqslant r$.

Definition 4.6. Let G be a bridgeless graph. The *circular flow index* of G, denoted by $\phi(G)$, is the smallest rational number r such that G admits a nowhere-zero circular r-flow.

It is proved in [17] that the number r in Definition 4.6 indeed exists. The following theorem was proved in [15] as an approach to Conjecture 1.2 (and Conjecture 1.3).

Theorem 4.7. *(Galluccio and Goddyn [15], also see [40].) For every* 6*-edge-connected graph G, $\phi(G) < 4$.*

The following corollary of Theorem 1.7 is an improvement of Theorem 4.7.

Corollary 4.8. *For every* 6*-edge-connected graph G, $\phi(G) \leqslant 3$.*

In Section 4.4, Corollaries 4.5 and 4.8 are further strengthened where the edge-connectivity is replaced by odd-edge-connectivity.

4.3. Group connectivity

Group connectivity was introduced in [30] as a generalization of integer flows.

Definition 4.9. Let Γ be an abelian group. Let G be a graph.

(i) A mapping $\beta : V(G) \mapsto \Gamma$ is called a *Γ-boundary* of G if $\sum_{v \in V(G)} \beta(v) = 0$.

(ii) The graph G is *Γ-connected* if, for every Γ-boundary β, there is an orientation D_β and a mapping $f_\beta : E(G) \mapsto \Gamma - \{0\}$ such that for every vertex $v \in V(G)$,

$$\sum_{e \in E^+_{D_\beta}(v)} f_\beta(e) - \sum_{e \in E^-_{D_\beta}(v)} f_\beta(e) = \beta(v).$$

The following is the original (and equivalent) version of Conjecture 1.10(iii).

Conjecture 4.10. *(Jaeger, Linial, Payan and Tarsi [30].) Every* 5*-edge-connected graph is* Z_3*-connected.*

The following corollary of Theorem 1.12 is a partial result on Conjecture 4.10.

Corollary 4.11. *Every* 6*-edge-connected graph is* Z_3*-connected.*

4.4. Odd-edge-connectivity

The *odd-edge-connectivity* of a graph is defined as the size of a smallest edge-cut of odd size. For references and recent results on graph flow and odd-edge-connectivity, see e.g. [51].

The 3-flow conjecture (Conjecture 1.2) by Tutte was originally proposed for odd-5-edge-connected graphs (see Open Problem # 97 and Conjecture 21.16 in [7]).

Inspired by Tutte's original conjecture for odd-edge-connectivity, we will further extend some of main results by replacing edge-connectivity by odd-edge-connectivity.

The results of Thomassen [57], when restricted to modulo k-orientation, remain valid under the weaker condition that edge-connectivity is replaced by odd-edge-connectivity. So does the main result of the present paper (and some of its consequences), as we now show. As the proof is repetition of earlier arguments, we leave some details for the reader.

Theorem 4.12. *Let k be an odd natural number and let G be an odd-$(3k-2)$-edge-connected graph. Assume that the edges of a smallest odd cut are oriented such that the cut is balanced modulo k, that is, the number of edges oriented in one direction is congruent (modulo k) to the number of edges oriented in the other direction. Then the pre-orientation of this cut can be extended to a modulo k-orientation of G.*

Proof. The proof is by induction on the number of edges of G.

We may assume that one side of the pre-oriented cut is a single vertex z_0. For otherwise, we apply induction to each side of the cut after contracting the other side.

We claim that for any vertex set $A \subseteq V(G) - z_0$ such that $1 < |A| < |V(G) - z_0|$ and $d(A)$ is odd, we have $d(A) \geqslant 3k$. For, if $d(A) \leqslant 3k-2$, then by a proof similar to that of Claim 1, we apply induction to G/A and then to G/A^c.

Then by a proof similar to that of Claim 2, we claim that there is no vertex of G having even degree, or equivalently all vertices of G have odd degree.

Now G must have a vertex set $A \subseteq V(G) - z_0$ such that $d(A) \leqslant 2k - 2$ and is even. For otherwise G satisfies the conditions of Theorem 3.1 and Theorem 4.12 follows.

Choose A to be minimal. We contract A and use induction. Then we contract A^c and by the minimality of A we can apply Theorem 3.1 to the graph G/A^c. $\square$

Motivated by some early results on odd-edge-connectivity and some conjectures by Jaeger (such as, Conjectures 1.10(i) and 1.3), the following problems were proposed in [69].

Conjecture 4.13. *(See [69].)*

(i) *For each positive integer p, every odd-$(4p+1)$-edge-connected graph admits a nowhere-zero circular $(2+\frac{1}{p})$-flow;*

(ii) *For each positive integer p, there is a function $f(p)$ such that every graph with odd-edge-connectivity $f(p)$ admits a nowhere-zero circular $(2+\frac{1}{p})$-flow.*

Conjecture 4.13(ii) is now solved by Theorem 4.12. Specifically, we have the following corollary.

Corollary 4.14. *For each positive integer p, every odd-$(6p+1)$-edge-connected graph admits a nowhere-zero circular $(2+\frac{1}{p})$-flow.*

Corollary 4.14 is a partial result on Conjecture 4.13(i) and extends Corollary 4.5 where the edge-connectivity is replaced by odd-edge-connectivity.

For the special case of $k=3$, we have the following strengthening of Theorem 1.7.

Corollary 4.15. *If G is a graph of odd-edge-connectivity at least 7, then G admits a nowhere-zero 3-flow (that is, the circular flow index $\phi(G) \leqslant 3$).*

Corollary 4.15 also improves two recent results in [51], namely

(1) *every graph G of odd-edge-connectivity at least $(4\lceil \log|V(G)|\rceil + 1)$ admits a nowhere-zero 3-flow*, and

(2) *every graph G of odd-edge-connectivity at least 7 has circular flow index $\phi(G) < 4$.*

5. Concluding remarks

5.1. τ-Connectivity and the sharpness of the main result

Suppose G is a graph with a Z_k-boundary β. Let A be a vertex subset of $V(G)$. By the definition of τ in Section 2.1, we know that $d(A) - |\tau(A)|$ is always an even integer. We say the τ-connectivity of G with a boundary is $h(k)$ if $d(A) - |\tau(A)| \geqslant h(k)$ for every proper, nonempty vertex subset $A \subset V(G)$, where $h(k)$ is the largest possible integer function of k.

Thomassen proved in [57] that a graph G with a boundary β has a β-orientation if the τ-connectivity of G is $2k^2$ (and $2k^2$ can be replaced by 6 for $k=3$). In the present paper it is shown that $h(k) = 2k-2$ suffices. This is the best possible in the sense that $2k-2$ cannot be replaced by $2k-4$. Indeed, it is not difficult to check that the τ-connectivity of the cartesian product $K_k * K_{k-1}$ with boundary $\beta = 1$ is $2k-4$ (because $\tau = 1$), but it has no β-orientation.

For $k=3$ the sharpness of our main result is also demonstrated by several families of graphs described in [30,38,41]. Note that some of these are planar or triangular (meaning that every edge is in a triangle). Planar graphs are obviously of interest in connection with flow problems. Triangular graphs have also received some attention in this context, see e.g. [14,34,37].

5.2. Modulo k-orientation for even k

Definition 5.1. Let G be a graph, let k be an integer, $k \geqslant 3$, and let $\theta : V(G) \mapsto Z_k$ be a mapping such that $\sum_{v\in V(G)} \theta(x) \equiv |E(G)| \pmod k$. An orientation D of G is called a θ-orientation if, for every vertex $x \in V(G)$,

$$d^+(x) \equiv \theta(x) \pmod k.$$

For each odd positive integer k, Theorem 1.11 is equivalent to the following, which is Theorem 2 in [57] (and which also holds for even k):

Theorem 5.2. *(Thomassen [57].) Let G be a graph, let k be an integer, $k \geqslant 2$, and let $\theta : V(G) \mapsto Z_k$ be a mapping such that $\sum_{v\in V(G)} \theta(x) \equiv |E(G)| \pmod k$. If G is $(2k^2+k)$-edge-connected, then G has a θ-orientation.*

This theorem has applications also to graph decompositions [58].

Theorem 5.2 was proved also for k even. The proof of the main theorem in the present paper can be modified so that the bound on the edge-connectivity in Theorem 5.2 can be lowered to $3k-2$, also for k even.

Lowering the bound $3k-3$ in Theorem 1.12 to $3k-4$ would prove the 3-flow conjecture. It cannot be lowered to $3k-6$ when $k=3$. But maybe it can be lowered to $3k-6$ for $k \geqslant 5$. If true, that would prove the 5-flow conjecture.

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