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Flows and parity subgraphs of graphs with large odd-edge-connectivity

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ABSTRACT

The odd-edge-connectivity of a graph G is the size of the smallest odd edge cut of G. Tutte conjectured that *every odd-5-edge-connected graph admits a nowhere-zero 3-flow*. As a weak version of this famous conjecture, Jaeger conjectured that *there is an integer k such that every k-edge-connected graph admits a nowhere-zero 3-flow*. Jaeger [F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory Ser. B 26 (1979) 205–216] proved that *every 4-edge-connected graph admits a nowhere-zero 4-flow*. Galluccio and Goddyn [A. Galluccio, L.A. Goddyn, The circular flow number of a 6-edge-connected graph is less than four, Combinatorica 22 (2002) 455–459] proved that *the flow index of every 6-edge-connected graph is strictly less than 4*. This result is further strengthened in this paper that *the flow index of every odd-7-edge-connected graph is strictly less than 4*. The second main result in this paper solves an open problem that *every odd-$(2k+1)$-edge-connected graph contains k edge-disjoint parity subgraphs*. The third main theorem of this paper proves that *if the odd-edge-connectivity of a graph G is at least $4\lceil \log_2 |V(G)|\rceil + 1$, then G admits a nowhere-zero 3-flow*. This result is a partial result to the weak 3-flow conjecture by Jaeger and improves an earlier result by Lai et al. The fourth main result of this paper proves that *every odd-$(4t+1)$-edge-connected graph G has a circular $(2t+1)$ even subgraph double cover*. This result generalizes an earlier result of Jaeger.

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1. Introduction

All graphs considered in this paper may contain parallel edges or loops.

The odd-edge-connectivity of a graph G, denoted by $\lambda_o(G)$, is the size of a smallest odd edge cut of G. For graphs with large odd-edge-connectivity, small edge cuts (of even size) may still exist. It is evident that odd-edge-connectivity plays a more important role than edge-connectivity in the study of some problems related to flow and even subgraph covers. However, there are not many results or methods developed yet to deal with small even edge cuts.

In this paper, we develop some methods to deal with small even cuts. For even degree vertices, the vertex splitting method (see Definition 4.8 and Lemma 4.9) is applied and the odd-edge-connectivity is preserved. For nontrivial small even cuts, contractions of k-tree blocks as contractible configurations (see Definitions 4.4 and 2.4) are applied. With those new methods, some earlier results in those areas are extended from λ-edge-connected graphs to odd-λ-edge-connected graphs (main results: Theorems 3.4, 3.7 and 3.11, Corollary 3.13).

The rest of the paper is constructed as follows. Notation and definitions are in Section 2. In Section 3, we show the main results of the paper. In Section 4 we provide lemmas used to prove the main theorems. Sections 5–8 are devoted to the proofs of Theorems 3.4, 3.7, 3.11, and Corollary 3.13.

2. Notation

The basic graph theory concepts and notation in this paper follow those of [3,9]. For background reading on flows, etc. see [36,18,39].

A **circuit** is a connected 2-regular subgraph. A graph/subgraph H is **even** if $d_H(v)$ is even for every $v \in V(H)$ (an even subgraph is also called a cycle in many literatures related to this area).

Let $G = (V, E)$ be an undirected graph and X be a nonempty proper subset of $V(G)$. The set of all edges between X and $Y = V(G) - X$, denoted by (X, Y), is an edge cut of G. If G is a directed graph under an orientation D, then the set of arcs from X to Y is denoted by $[X, Y]_D$.

An edge cut is *odd* if it contains an odd number of edges.

Definition 2.1. A graph G is **odd-$(2k+1)$-edge-connected** if every odd edge cut contains at least $(2k+1)$ edges. The **odd-edge-connectivity** of G, denoted by $\lambda_o(G)$, is the size of a smallest odd edge cut of G.

Definition 2.2. (i) Contracting an edge e of a graph G is to delete the edge and then (if the edge is not a loop) identify its ends. The resulting graph is denoted by G/e. (Note that G/e may generate loops or parallel edges if $e = xy$ with x and y connected by parallel edges or having common neighbors.)

(ii) Let $F \subseteq E(G)$. The graph obtained from G by contracting every edge of F is denoted by G/F.

(iii) Let $\{H_1, \ldots, H_t\}$ be a set of vertex-disjoint, connected subgraphs of G. The graph obtained from G by contracting every $E(H_i)$ is denoted by $G/\{H_1, \ldots, H_t\}$.

Definition 2.3. Let $\{v_1, v_2, \ldots, v_t\}$ be the set of vertices of degree 2 of a graph G, and denote the edges incident with v_i by e_i, e_i', for $i = 1, \ldots, t$. Then $\overline{G} = G/\{e_1, e_2, \ldots, e_t\}$ is called the **suppressed graph** of G.

Let F_1 and F_2 be two subgraphs of G, the *symmetric difference* of F_1 and F_2, denoted by $F_1 \Delta F_2$, is the subgraph of G induced by the edge set $[E(F_1) \cup E(F_2)] - [E(F_1) \cap E(F_2)]$. Let $F_1, F_2, \ldots, F_t$ be subgraphs of G. We use $\Delta_i F_i$ to denote the subgraph of G induced by the edges that appear an odd number of times in $\{E(F_1), \ldots, E(F_t)\}$.

Definition 2.4. A graph H is a **contractible configuration for a given property** $\mathcal{P}$ if, for any graph G containing H as a subgraph, G has property $\mathcal{P}$ if and only if G/H has property $\mathcal{P}$.

2.1. Flows

Definition 2.5. Let $G = (V, E)$ be a graph. An ordered pair (D, f) is an **integer flow** of G if D is an orientation of $E(G)$ and $f : E(G) \to Z$, the set of integers, such that the total in-flow equals the total out-flow at every vertex. An integer flow (D, f) is a k-**flow** if $|f(e)| \leqslant k - 1$ for every edge e of G. It is **nowhere-zero** if $f(e) \neq 0$ for every edge e of G. An integer k-flow (D, f) of a graph G is **positive** if $f(e) > 0$ for every edge e of a graph G.

Circular flow, introduced in [12], is a real line extension of integer flow problems. (See [39], for a comprehensive survey on this subject, and see [7,30,27], etc. for related results.)

Definition 2.6. Let $G = (V, E)$ be a graph. An ordered pair (D, f) is a **circular flow** of G if D is an orientation of $E(G)$ and $f : E(G) \to Q$, the set of rational numbers, such that the total in-flow equals the total out-flow at every vertex. A circular flow (D, f) is a **nowhere-zero** r**-flow** if $f(e) \neq 0$ for every $e \in E(G)$ and $\frac{\max_{e \in E(G)}\{|f(e)|\}}{\min_{e \in E(G)}\{|f(e)|\}} \leqslant r - 1$.

Definition 2.7. The **flow index** $\phi(G)$ of a graph G is the smallest rational number r such that G admits a nowhere-zero r-flow.

Note that such an r exists by Lemma 4.3. The following lemma can be viewed as an alternative definition of flow index.

Lemma 2.8. *(See Goddyn, Tarsi and Zhang [12].) Let $\mathcal{D}$ be the set of all orientations of G, and let (A, B) be any edge cut of G. Then*

$$\phi(G) = \min_{D \in \mathcal{D}} \left\{ \max_{(A,B)} \frac{|[A, B]_D|}{|[B, A]_D|} \right\} + 1.$$

For a flow (D, f) of a graph G, let

$$E_{f=h} = \left\{ e \in E(G)\colon\ f(e) = h \right\}.$$

2.2. Even subgraph covers

Definition 2.9. Let G be a graph. A family $\mathcal{F}$ of even subgraphs of G is an **even subgraph cover** of G if every edge of G is contained in at least one member of $\mathcal{F}$. $\mathcal{F}$ is called a **double cover** if each edge of G is contained in precisely two members of $\mathcal{F}$.

Definition 2.10. An even subgraph double cover $\mathcal{F} = \{C_0, C_1, \ldots, C_{r-1}\}$ of a graph G is called a **circular** r **even subgraph double cover** of G if $E(C_i) \cap E(C_j) \neq \emptyset$ if and only if $|j - i| \leqslant 1 \pmod r$.

3. Main results

3.1. Flows and flow index

Integer flow was originally introduced by Tutte [32,33] as a generalization of map coloring problems. One major open problem in the area of integer flow is the following conjecture.

Conjecture 3.1. *(Tutte, Open Problem # 97 and Conjecture 21.16 in [3], p. 157 of [9], or Conjecture 1.1.8 in [36].) Every graph with odd-edge-connectivity at least* 5 *admits a nowhere-zero* 3*-flow.*

It is pointed out in [29,17] that a 2-edge-cut does not exist in any smallest counterexample to some well-known flow conjectures and circuit cover conjectures. The 3-flow conjecture (Conjecture 3.1) by Tutte [3] was originally proposed for odd-5-edge-connected graphs. Kochol [20] proved that minimal counterexamples to the conjecture contain no 4-edge-cuts.

The following are two of the best partial results to Conjecture 3.1.

Theorem 3.2. *(See Jaeger [15].) Every* 4*-edge-connected graph admits a nowhere-zero* 4*-flow.*

Theorem 3.3. *(See Galluccio and Goddyn [11].) If G is a* 6*-edge-connected graph, then the flow index $\phi(G) < 4$.*

Theorem 3.3 is further generalized in this paper.

Theorem 3.4. *If G is an odd-*7*-edge-connected graph, then its flow index $\phi(G) < 4$.*

3.2. Nowhere-zero 3-flows

A weak version of Conjecture 3.1 was proposed by Jaeger.

Conjecture 3.5. *(See Jaeger [15].) There is an integer h such that every h-edge-connected graph admits a nowhere-zero* 3*-flow.*

The following is an approach to Conjecture 3.5.

Theorem 3.6. *(See Lai and Zhang [23].) Every $4\lceil \log_2 n \rceil$-edge-connected graph with n vertices admits a nowhere-zero* 3*-flow.*

Theorem 3.6 is further generalized in this paper.

Theorem 3.7. *Every odd-$(4\lceil \log_2 n \rceil + 1)$-edge-connected graph with n vertices admits a nowhere-zero* 3*-flow.*

3.3. Parity subgraphs

Definition 3.8. (See [5].) Let $G = (V, E)$ be a graph and $H = (V, E')$ be a spanning subgraph of G (where $E' \subseteq E$). The subgraph H is a **parity subgraph** of G if, for every vertex $v \in V(G)$, $d_G(v) \equiv d_H(v) \pmod 2$.

It was proved by Tutte [34] and Nash-Williams [26] that *every* 2*k-edge-connected graph contains at least k edge-disjoint spanning trees*, and proved by Itai and Rodeh [14] that *every spanning tree of a graph G contains a parity subgraph*. The following theorem is an immediate corollary of these results.

Theorem 3.9. *Every* 2*k-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

It is well known that the search of parity subgraphs plays a central role in the proofs of some important theorems in the area of integer flows. For example, the 4-flow theorem (Theorem 3.2) is proved by Jaeger [15] with the following approach.

Theorem 3.10. *(See Jaeger [15].) If a graph G contains two edge-disjoint parity subgraphs, then G admits a nowhere-zero* 4*-flow.*

The 8-flow theorem (Jaeger [15]) was proved with a similar approach: Let G be a 2-edge-connected graph and $2G$ be the graph obtained from G by doubling every edge. It can be proved that $2G$ contains three edge-disjoint parity subgraphs, and therefore, G admits a nowhere-zero 8-flow.

Theorem 3.9 is generalized in this paper for odd-edge-connectivity and, therefore, solves an open problem proposed in [35,6,36].

Theorem 3.11. *Every odd-$(2k+1)$-edge-connected graph G contains at least k edge-disjoint parity subgraphs of G.*

3.4. Circular even subgraph double covers

The concept of circular even subgraph double cover, as defined in Definition 2.10, was introduced by Jaeger [16].

Theorem 3.12. *(See Jaeger [16].) Let t be an integer. Every $(4t)$-edge-connected graph has a circular $(2t)$ even subgraph double cover.*

Theorem 3.12 can be generalized by applying Theorem 3.11 to get the following corollary.

Corollary 3.13. *Every odd-$(4t+1)$-edge-connected graph G has a circular $(2t+1)$ even subgraph double cover.*

4. Lemmas

4.1. Circuit decomposition

Lemma 4.1. *Let G be an even graph containing a spanning tree T and $A \subseteq E(G) - E(T)$ with $|A| = 2$. Then G has a circuit decomposition $\mathcal{C} = \{C_1, \dots, C_\mu\}$, where each C_i is a circuit, such that edges of A belong to distinct members of $\mathcal{C}$.*

Proof. Let $A = \{e_1, e_2\}$, and let C_1 be the circuit contained in $T + e_1$. Note that $G' = G - E(C_1)$ remains even and $e_2 \in E(G')$. Hence, G has a circuit decomposition $\{C_1, C_2, \dots, C_\mu\}$ where $\{C_2, \dots, C_\mu\}$ is a circuit decomposition of the even subgraph G'. $\square$

4.2. Flows and orientations

The following lemma will be used in the proofs of some results and can be found in many textbooks (such as [36]).

Lemma 4.2. *An ordered pair (D, f) is a flow of a graph G if and only if*

$$\sum_{e \in [A,B]_D} f(e) = \sum_{e \in [B,A]_D} f(e)$$

for every edge cut (A, B) of G.

The following lemma describes the relation between flows and orientations.

Lemma 4.3. *(Hoffman [13] or see [2].) Let G be a bridgeless graph, D an orientation of G and a, b two positive integers ($a \leqslant b$). The following statements are equivalent.*

(1) $$\frac{a}{b} \leqslant \frac{|[A,B]_D|}{|[B,A]_D|} \leqslant \frac{b}{a}$$

for every edge cut (A, B) of G.

(2) *G admits a nowhere-zero integer flow (D, f_1) such that $a \leqslant f_1(e) \leqslant b$ for every $e \in E(G)$.*

4.3. k-Tree blocks

The following is one of the key concepts frequently used in the proofs of this paper.

Definition 4.4. Let $G = (V, E)$ be a graph and $V' \subseteq V(G)$. Let H be the subgraph of G induced by V'. The induced subgraph H is k-**tree connected** if it contains k edge-disjoint spanning trees (of H). A k-tree connected subgraph is **maximal** if, for every k-tree connected subgraph H' other than H, $V(H)$ is not a proper subset of $V(H')$. A maximal k-tree connected subgraph is called a k-**tree block**.

The following is the key lemma of this paper and will be used in the proof of Theorems 3.4, 3.7 and 3.11.

Lemma 4.5. *(See Lemma 2.4.3 in [9] and [38].) Let $\{R_1, R_2, \dots, R_k\}$ be a set of edge-disjoint spanning forests of a graph G with $|\bigcup_i E(R_i)|$ as large as possible. If there is an edge $e \in E(G) - \bigcup_{i=1}^k E(R_i)$, then there is a k-tree block H of G containing e.*

A k-tree block H is **trivial** if $|V(H)| = 1$. All k-tree blocks in this paper are nontrivial unless otherwise stated.

By counting the numbers of edges needed for k edge-disjoint spanning trees, we can easily get the following lemma.

Lemma 4.6. *(See Nash-Williams [26].) If $\delta(G) \geqslant 2k$, then there is a nontrivial k-tree block of G.*

Proof. The number of edges in a spanning forest is at most $|V(G)| - 1$. Let G be a graph with $\delta(G) \geqslant 2k$. Then, for any set of k edge-disjoint spanning forests, there must be some edge e that is not contained in their union since $|E(G)| \geqslant k|V(G)| > k(|V(G)| - 1)$. This lemma follows immediately by applying Lemma 4.5. $\square$

Lemma 4.7. *(Catlin [4], also see [22].) Let G be a graph which is not k-tree connected, and let $\{H_1, H_2, \dots, H_t\}$ be the set of all k-tree blocks of G. Then*

(i) *H_i and H_j are vertex-disjoint if $i \neq j$;*
(ii) *$G' = G/\{H_1, H_2, \dots, H_t\}$ is not k-tree connected, and contains no k-tree blocks.*

4.4. Vertex splitting

Definition 4.8. Let v be a vertex of a graph G and let $e_1, e_2 \in E(v)$ where e_1 is incident with v and x, e_2 is incident with v and y. The graph obtained from G by deleting the edges e_1 and e_2 and adding a new edge e_3 joining x and y, denoted by $G_{[v;\{vx,vy\}]}$, is called **the graph obtained from** G **by splitting two edges** $e_1 = vx$ **and** $e_2 = vy$ **away from** v.

Vertex splitting is one of the commonly used techniques in areas of integer flows and even subgraph covers.

This operation was originally introduced by Fleischner [10]. By applying Fleischner's splitting lemma, one is able to prove that *the smallest counterexample to the even subgraph double cover conjecture* [28,31] *is cubic* [17], *and the smallest counterexample to the* 5*-flow conjecture* [33] *is also cubic* [15]. Furthermore, *the* 6*-flow theorem was proved by first reducing the problem to being cubic* [29].

Fleischner's splitting lemma (which preserves the property of being bridgeless) was further generalized by Mader [25] for preserving edge-connectivity, and generalized by Zhang for preserving the odd-edge-connectivity (Lemma 4.9).

Lemma 4.9. *(See Zhang [37].) Let $G = (V, E)$ be a graph with odd-edge-connectivity λ_o. Assume that there is a vertex $v \in V(G)$ with degree d such that $d \notin \{2, \lambda_o\}$. Arbitrarily label the edges incident with v as $\{e_1, e_2, \dots, e_d\}$, then there is an integer $i \in \{1, 2, \dots, d\}$ such that the odd-edge-connectivity of the new graph $G_{[v;\{e_i,e_{i+1}\}]}$ obtained from G by splitting e_i and e_{i+1} $(\mathrm{mod}\ d)$ away from v remains λ_o.*

5. Proof of Theorem 3.4

Definition 5.1. Let (D, f) be a positive 4-flow of a graph G. An edge cut (X, Y) is **bad** if $f(e) = 1$ for every edge $e \in [X, Y]_D$ and $f(e) = 3$ for every edge $e \in [Y, X]_D$.

The following lemma is proved by applying Lemmas 4.2, 4.3 and 2.8.

Lemma 5.2. *Let G be a graph with $\phi(G) \leqslant 4$. Then $\phi(G) = 4$ if and only if, for every positive 4-flow (D, f) of G, G must have a bad cut.*

Proof. Suppose $\phi(G) \leqslant 4$. By Lemma 2.8, $\phi(G) = 4$ is equivalent to

$$\text{if } \max_{(A,B)} \frac{|[A, B]_D|}{|[B, A]_D|} \leqslant 3, \quad \text{then } \max_{(A,B)} \frac{|[A, B]_D|}{|[B, A]_D|} = 3,$$

where the maximum is taken over all edge cuts (A, B). By Lemma 4.3, this is equivalent to the following statement

for every positive 4-flow (D, f), there is some edge cut (A, B) such that $|[A, B]_D| = 3|[B, A]_D|$.

By Lemma 4.2, this is equivalent to the following statement

for every positive 4-flow (D, f), there is some edge cut (A, B) such that $[A, B]_D \subset E_{f=1}$ and $[B, A]_D \subset E_{f=3}$,

which, by definition, is precisely

"*every positive 4-flow has a bad cut.*" □

The following lemma is an immediate corollary of Lemma 5.2.

Lemma 5.3. *(See Lai et al. [24].) Let (D, f) be a nowhere-zero 4-flow of a graph G. If $E_{f=\pm 2}$ contains a spanning tree of G, then $\phi(G) < 4$.*

Both Lemmas 5.2 and 5.3 are used in the proof of Theorem 3.4.

The following lemmas are used in the proof of the main theorem whenever a vertex splitting occurs or a 2-edge-cut exists. Note that Lemma 5.4 can be easily proved.

Lemma 5.4. *If G' is a graph obtained from G by splitting a pair of incident edges, then $\phi(G') < 4$ implies that $\phi(G) < 4$.*

Lemma 5.5. *(See Seymour [29].) Assume that $\{e, e'\}$ is a 2-edge-cut of a graph G. Let G' be the graph obtained from G by contracting the edge e. Then $\phi(G') = \phi(G)$.*

Let e be an edge of a graph G. Every odd edge cut of G/e is an odd edge cut of G, therefore, $\lambda_o(G/e) \geqslant \lambda_o(G)$ and we have the following observation.

Observation 5.6. The operation of edge contraction does not decrease odd-edge-connectivity.

The following lemma is used to construct positive 4-flows in the proof of Theorem 3.4.

Lemma 5.7. *Let G be a graph with two edge-disjoint spanning trees T_1, T_2, and A_1, A_2 be two disjoint edge sets both distinct from T_1 and T_2. Let $\{e_1, e_2\} \subset A_1$, and $\{e_3, e_4\} \subset A_2$. For any orientation D^0 of $\{e_1, e_2, e_3, e_4\}$, there is a positive 4-flow (D, f) such that*:

(i) *The orientations of D and D^0 coincide on e_j, for $1 \leqslant j \leqslant 4$;*
(ii) *$f(e) = i$ for $e \in A_i$, $1 \leqslant i \leqslant 2$.*

Proof. Without loss of generality, we may assume $A_1 = E(G) \setminus \{T_1, T_2, A_2\}$.

Let $C_{e,i}$ be the circuit contained in $T_i + e$ for an edge $e \notin T_i$. Let

$$C_1 = \Delta_{e \in A_1 \cup T_1} C_{e,2} \quad \text{and} \quad C_2 = \Delta_{e \in A_2 \cup T_2} C_{e,1}.$$

It is easy to see that C_1 and C_2 are even subgraphs, $A_1 \cup T_1 \subset C_1$, $A_2 \cup T_2 \subset C_2$, and $A_1 \cap C_2 = A_2 \cap C_1 = \emptyset$.

Now A_1 and T_1 are disjoint subsets of the even graph C_1. By Lemma 4.1, there is a circuit decomposition of C_1 such that e_1 and e_2 are in different circuits. Extend the orientations of e_1 and e_2 under D^0 to all edges of C_1 so that each member of the circuit decomposition is a directed circuit. Let (D_1, f_1) be the non-negative 2-flow of G with support C_1 agreeing with the orientations of edges in the directed circuits.

Similarly, there is a non-negative 2-flow (D_2, f_2) of G with support C_2, such that the orientation of e_3, e_4 on D_2 are the same as on D^0.

Let D_3 be the orientation of G obtained from D_1 and D_2 that preserves the orientation on C_2 and $C_1 \setminus C_2$. Note that the orientations of e_1, e_2, e_3, e_4 remain the same since each D_i agrees with each e_j if it is in C_i. Let (D_3, f_4) be the 2-flow obtained from (D_1, f_1) by reversing orientations of edges $e \in C_1 \cap C_2$ if the orientations of D_1 and D_2 disagree at e and changing signs of the flow weight. That is,

$$f_4(e) = \begin{cases} -f_1(e) & \text{if the orientations of } D_1 \text{ and } D_2 \text{ disagree at } e, \\ f_1(e) & \text{otherwise} \end{cases}$$

for every edge $e \in C_1$.

It is obvious that $(D_3, f_3 = f_4 + 2f_2)$ is a positive 4-flow of G and $f_3(e) = i$ for each $e \in E(A_i)$. □

Proof of Theorem 3.4. By way of contradiction, let G be an odd-7-edge-connected graph such that $\phi(G) \geqslant 4$ with the least number of edges and vertices.

I. By Lemma 5.5 and Observation 5.6, the graph G does not have 2-edge-cuts. Therefore, the edge-connectivity of G is at least 4.

II. If G has three edge-disjoint spanning trees T_1, T_2, T_3, then by Lemma 5.7, letting $T_3 = A_2$, we have a positive 4-flow (D, f) such that $T_3 \subseteq E_{f=2}$. By Lemma 5.3, $\phi(G) < 4$, contradicting the choice of G. Therefore G is not 3-tree connected.

III. We claim that $\delta(G) \geqslant 7$. Assume that there is a vertex v with degree at most 6. By Lemma 4.9, we can split a pair of edges away from v to get a smaller odd-7-edge-connected graph G'. By the minimality of G, $\phi(G') < 4$. By Lemma 5.4, $\phi(G) < 4$ as well, which is a contradiction.

IV. By III and Lemma 4.6, G contains some 3-tree blocks. Let $\{H_1, H_2, \ldots, H_a\}$ be the collection of all 3-tree blocks of G and let $G^\star = G/\{H_1, \ldots, H_a\}$. By II, G is not 3-tree connected, so Lemma 4.7(ii) tells us that $G^\star$ does not contain a 3-tree block. Hence $\delta(G^\star) \leqslant 5$, by Lemma 4.6. Now $\delta(G^\star)$ cannot be 1, 3, or 5 since $G^\star$ is 7-odd-edge-connected as contraction does not decrease odd-edge-connectivity (by Observation 5.6). Part I tells us that $\delta(G^\star)$ cannot be 2 and so it must be 4.

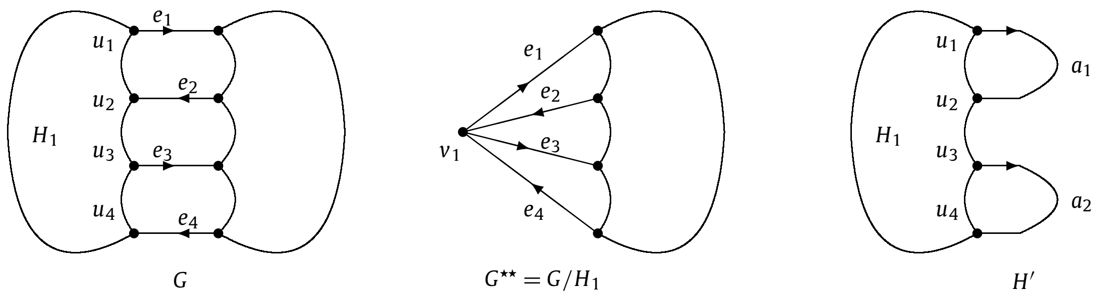

Fig. 1.

V. By IV, let v_1 be a vertex in $G^\star$ with degree 4. Let H_1 be the 3-tree block of G corresponding to v_1 (the small degree vertex v_1 is created by the contraction of H_1 since, by III, $\delta(G) \geqslant 7$). Since the smaller graph $G^{\star\star} = G/H_1$ remains odd-7-edge-connected, its flow index is less than 4. By Lemma 5.2, let (D_0, f_0) be a positive 4-flow of G/H_1 with no bad cuts. Hence, under the orientation D_0,

$$\left|E^+(v_1)\right| = \left|E^-(v_1)\right| = 2.$$

(Here, $E^+(v_1)$ is the set of all arcs with tails at v_1, and $E^-(v_1)$ is the set of all arcs with heads at v_1.) Let $E(v_1) = \{e_1, e_2, e_3, e_4\}$.

VI. *Case one*: There is a pair of edges, say e_1, e_2 of $E(v_1)$ with $f_0(e_1) = f_0(e_2)$ and oriented in opposite orientation under D_0. Without loss of generality, we may assume that

$$E^+(v_1) = \{e_1, e_3\} \quad \text{and} \quad E^-(v_1) = \{e_2, e_4\},$$
$$f_0(e_1) = f_0(e_2) = w_1 \quad \text{and} \quad f_0(e_3) = f_0(e_4) = w_2$$

with $3 \geqslant w_1 \geqslant w_2 \geqslant 1$.

Let u_i be the endvertex of e_i in H_1, for $i = 1, 2, 3, 4$. (The vertices u_1, u_2, u_3 and u_4 may not be distinct.) Let H' be the graph obtained from H_1 by adding two arcs a_1 and a_2 such that a_1 joins u_1 to u_2 and a_2 joins u_3 to u_4. (See Fig. 1.) If $w_i = 1$, denote a_i by a_i^1; if $w_i = 2$, denote a_i by a_i^2; if $w_i = 3$, duplicate a_i and denote the edges by a_i^1 and a_i^2.

Since H_1 is a 3-tree block, it has three spanning trees T_1, T_2, T_3. Let $A_2 = T_3 \cup \{a_i^2\}$, and $A_1 = E(H') - E(T_1) - E(T_2) - E(A_2)$. By Lemma 5.7, H' admits a positive 4-flow (D_1, f_1) such that $E(T_3) \subset E_{f_1=2}$, and $f_1(a_i^j) = j$. Let (D, f) be an ordered pair of the entire graph G that the orientation D agrees with D_0 for edges in $E(G^{**}) = E(G) - E(H_1)$ and agrees with D_1 for edges in $E(H_1)$, and let

$$f(e) = \begin{cases} f_0(e) & \text{if } e \in E(G) - E(H_1), \\ f_1(e) & \text{if } e \in E(H_1) \end{cases}$$

for every edge $e \in E(G)$.

It is easy to check that (D, f) is a positive 4-flow.

VII. *Case two*: For every pair $\{e_i, e_j\} \subseteq E(v_1)$ with opposite orientations,

$$f_0(e_i) \neq f_0(e_j).$$

Without loss of generality, we assume that

$$E^+(v_1) = \{e_1, e_2\} \quad \text{and} \quad E^-(v_1) = \{e_3, e_4\},$$
$$f_0(e_1) = f_0(e_2) = 2, \qquad f_0(e_3) = 1 \quad \text{and} \quad f_0(e_4) = 3.$$

Let u_i be the endvertex of e_i in H_1, for $i = 1, 2, 3, 4$. (The vertices u_1, u_2, u_3 and u_4 may not be distinct.) Let H'' be the graph obtained from H_1 by adding three arcs a_1, a_2 and a_3 such that a_1 joins u_1 to u_3, a_2 joins u_2 to u_4 and a_3 joins u_3 to u_4. (See Fig. 2.)

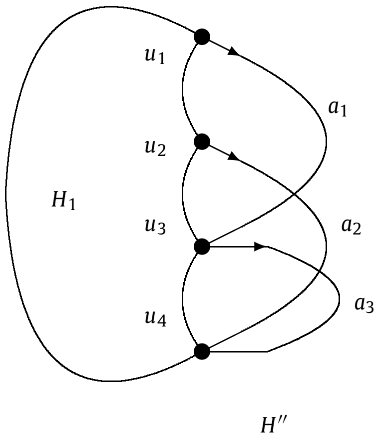

Fig. 2.

Since H_1 is a 3-tree block, it has three spanning trees T_1, T_2 and T_3. Let $A_2 = T_3 \cup \{a_1, a_2\}$, and $A_1 = E(H'') - E(T_1) - E(T_2) - E(A_2)$. By Lemma 5.7, there is a positive 4-flow (D_1, f_1) such that $E(T_3) \subset E_{f_1=2}$ and $f_1(a_1) = f_1(a_2) = 2$, $f_1(a_3) = 1$.

Let (D, f) be an ordered pair of the entire graph G that the orientation D agrees with D_0 for edges in $E(G^{**}) = E(G) - E(H_1)$ and agrees with D_1 for edges in $E(H_1)$, and let

$$f(e) = \begin{cases} f_0(e) & \text{if } e \in E(G) - E(H_1), \\ f_1(e) & \text{if } e \in E(H_1) \end{cases}$$

for every edge $e \in E(G)$. It is easy to check that (D, f) is a positive 4-flow.

VIII. $\phi(G) < 4$ (for Cases 1 and 2). We only need to show G does not have any bad cuts with respect to (D, f) (by Lemma 5.2). Assume that Q is a bad cut of G with respect to (D, f). If $Q \cap E(H_1) = \emptyset$, then it is an edge cut of $G^{\star\star}$. By Part V it is not bad. If $Q \cap E(H_1) \neq \emptyset$, then $Q \cap H_1$ is an edge cut of H_1. Note that T_3 is a spanning tree of H_1 and $T_3 \subseteq E_{f_1=2}$. Hence, there is an edge $e \in Q \cap T_3 \subseteq E_{f=2}$. Q is not bad with respect to (D, f) since there exists an edge e in Q such that $f(e) = 2$.

Hence, the flow index of G is less than 4, contrary to the choice of G. $\square$

6. Proof of Theorem 3.7

Definition 6.1. Let D be an orientation of a graph G, let Γ be an Abelian group and let $f : D \to \Gamma$ be a map. The **boundary** of f is the map $\partial f : V(G) \to \Gamma$ where $\partial f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e)$ for each vertex $v \in V(G)$. G is said to be Γ-**connected** if for every $b : V(G) \to \Gamma$ with $\sum_{v\in V(G)} b(v) = 0$, there exists a nowhere-zero map $f : D \to \Gamma$ with boundary $\partial f = b$.

The concept of group connectivity was introduced by Jaeger, Linial, Payan and Tarsi in [19] as a generalization of integer flow problems. The following lemma shows that for nowhere-zero Z_3-flow problems, the concepts of contractible configuration (see Definition 2.4) and group connectivity are consistent with each other.

Lemma 6.2. *(See [8,21].) Let H be a Z_3-connected subgraph of a graph G. Then H is a contractible configuration for having a nowhere-zero 3-flow.*

The following two theorems were proved by Barát and Thomassen, and we will use the second one in our proof. Note these two theorems are also generalizations of Theorem 3.6.

Theorem 6.3. *(See Barát and Thomassen [1].) Every $4\lceil \log_2 n \rceil$-edge-connected graph with n vertices is Z_3-connected.*

Theorem 6.4. *(See Barát and Thomassen [1].) Let G be a graph with n vertices. If G has $2\lceil \log_2 n\rceil$ edge-disjoint spanning trees, then G is Z_3-connected.*

Proof of Theorem 3.7. By way of contradiction, suppose G is the minimum counterexample with respect to order and size. Let $\lambda_o(G) = 2k+1$ with $k \geqslant 2\lceil \log_2 n\rceil$ where $n = |V(G)|$.

If $\delta(G) = 2$, then the suppressed graph $\overline{G}$ is smaller than the minimum counterexample G. $\overline{G}$ admits a nowhere-zero 3-flow, so G does.

We claim that $\delta(G) > 2k$. Otherwise, suppose $d_G(v) \leqslant 2k$ for some vertex $v \in V(G)$. By Lemma 4.9, we can split 2 edges away from v to get G' where G' has less edges and remains odd-$(2k+1)$-edge-connected. By the minimality of G, G' admits a nowhere-zero 3-flow, so G does.

By the claim above and Lemma 4.6, there is a k-tree block H of G. Note that G/H is still odd-$(2k+1)$-connected, so it admits a nowhere-zero 3-flow.

Since H has k edge-disjoint spanning trees where $k \geqslant 2\lceil \log_2 n\rceil \geqslant 2\lceil \log_2 |V(H)|\rceil$, by Theorem 6.4, H is Z_3-connected. By Lemma 6.2, G admits a nowhere-zero 3-flow, a contradiction. □

7. Proof of Theorem 3.11

Definition 7.1. Let G be a graph and let $T: V(G) \to Z_2$ be a zero-sum mapping. A T-**join** is a subgraph Q of G such that

$$d_Q(v) \equiv T(v) \pmod 2$$

for every $v \in V(G)$.

It is obvious that parity subgraph is a special case of T-join if $T(v) \equiv d_G(v) \pmod 2$ for every $v \in V(G)$. In this section, we will consider the packing problem of parity subgraphs, and T-joins.

Definition 7.2. Let G be a graph and k be an integer. Let $\vec{T} = (T_1, \dots, T_k)$ be such that, for each $i \in \{1, \dots, k\}$, $T_i: V(G) \to Z_2$ is a zero-sum mapping. A **multi**-$\vec{T}$-**join packing** is a set of edge-disjoint subgraphs $\{P_1, P_2, \dots, P_k\}$ such that $d_{P_i}(v) \equiv T_i(v) \pmod 2$.

The following lemma is needed in the proof of Lemma 7.4.

Lemma 7.3. *If G contains k edge-disjoint spanning trees $R_1, \dots, R_k$, then, for any $\vec{T} = (T_1, \dots, T_k)$ that each T_i is a zero-sum mapping: $V(G) \to Z_2$, G has a multi-$\vec{T}$-join packing $\{P_1, \dots, P_k\}$ such that $P_i \subseteq R_i$.*

Proof. Let $R_1, \dots, R_k$ be k edge-disjoint spanning trees. For every i, let

$$S_i = \bigl\{v \in V(G) \bigm| T_i(v) \equiv 1\bigr\} \pmod 2.$$

Note that $|S_i|$ is even. Partition S_i into pairs. For each pair of vertices of S_i, there is a path in R_i. Let P_i be the symmetric difference of these paths, then P_i is a T_i-join and contained in R_i. Therefore, $\{P_1, \dots, P_k\}$ is a multi-$\vec{T}$-join packing. □

See Definition 2.4 for the definition of a contractible configuration.

Lemma 7.4. *A k-tree block is a contractible configuration for having k edge-disjoint parity subgraphs.*

Proof. Let H be a k-tree block of G. We show that *G contains k edge-disjoint parity subgraphs if and only if G/H contains k edge-disjoint parity subgraphs.*

$(\Rightarrow)$ If G has k edge-disjoint parity subgraphs $P_1, P_2, \dots, P_k$, then $P_1/H, P_2/H, \dots, P_k/H$ are k edge-disjoint parity subgraphs of G/H, where P_i/H is the subgraph of G/H spanned by $E(P_i)/E(H)$.

$(\Leftarrow)$ Let $\{P_1', \dots, P_k'\}$ be a set of k edge-disjoint parity subgraphs of G/H. Let G_i be the graph obtained from G by deleting edges of P_i'. For each $i \in \{1, 2, \dots, k\}$, let $S_i = \{v \in V(G)\colon d_{G_i}(v) \text{ is odd}\}$.

Obviously, $S_i \subseteq V(H)$ since P'_i is a parity subgraph of the contracted graph G/H. For each $i \in \{1,\dots,k\}$, define a zero-sum mapping $T_i : V(H) \mapsto Z_2$ with $T_i(v) = 1$ if $v \in S_i$ and 0 otherwise. By Lemma 7.3, there is a multi-$\vec{T}$-join $\{P''_1,\dots,P''_k\}$ of H (where $\vec{T} = (T_1,\dots,T_k)$). It is evident that $P'_1 \cup P''_1, P'_2 \cup P''_2, \dots, P'_k \cup P''_k$ are k edge-disjoint parity subgraphs of G. $\square$

Proof of Theorem 3.11. By way of contradiction, we assume that G is the minimum counterexample with respect to the cardinality of edges and vertices.

By Lemma 7.3, the minimum counterexample G is not a k-tree block.

If $\delta(G) = 2$, then the suppressed graph $\overline{G}$ is smaller than the minimum counterexample G. $\overline{G}$ contains k edge-disjoint parity subgraphs, so G does.

We claim that

$$\delta(G) \geqslant 2k. \tag{1}$$

Otherwise, there is a vertex v with degree at most $2(k-1)$. By Lemma 4.9, we can split a pair of edges away from v, and the resulting graph G' remains odd-$(2k+1)$-edge-connected. Note that (by Definition 4.8 for vertex splitting) $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)| - 1$. Hence, by the minimality of G, G' has k edge-disjoint parity subgraphs $P'_1, P'_2, \dots, P'_k$. If we reverse the operation of splitting, the corresponding subgraphs $P_1, P_2, \dots, P_k$ are also parity subgraphs of G, which contradicts the choice of G.

By inequality (1) and Lemma 4.6, there is a k-tree block H of G. Since G/H has less edges and satisfies the conditions of the theorem, G/H has k edge-disjoint parity subgraphs $P'_1, P'_2, \dots, P'_k$. By Lemma 7.4, G has k edge-disjoint parity subgraphs. This is a contradiction and completes the proof of the theorem. $\square$

Corollary 7.5. *Every odd-$(2k+1)$-connected graph G has a $(2\lfloor \frac{k}{2} \rfloor + 1)$ parity subgraph decomposition.*

Proof. By Theorem 3.11, let $P_1, P_2, \dots, P_k$ be edge-disjoint parity subgraphs of G and let $Q = G \setminus \bigcup_{i=1}^{k} E(P_i)$. If k is even, then Q is also a parity subgraph of G, thus we have a set of $k+1$ edge-disjoint parity subgraphs whose union is the entire graph G. If k is odd, then $k = (2\lfloor \frac{k}{2} \rfloor + 1)$ and Q is even. Thus, $\{P_1, \dots, P_{k-1}, P_k \cup Q\}$ is the parity subgraph decomposition of G. $\square$

8. Proof of Corollary 3.13

By Theorem 3.11, G has $2t$ edge-disjoint parity subgraphs $\{P_1, \dots, P_{2t}\}$. Let $P_0 = G \setminus \bigcup_{i=1}^{2t} E(P_i)$. Then P_0 is also a parity subgraph of G.

$$\left\{P_i \cup P_{i+1}\colon\ i = 0, 1, \dots, 2t \ (\mathrm{mod}\ 2t+1)\right\}$$

is a circular $(2t+1)$ even subgraph double cover of G. $\square$

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