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Cycle covers (I) – Minimal contra pairs and Hamilton weights

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ABSTRACT

Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. A *faithful circuit cover* of the pair (G, w) is a family of circuits in G which covers each edge e of G precisely $w(e)$ times. A circuit C of G is *removable* if the graph obtained from G by deleting all weight 1 edges contained in C remains bridgeless. A pair (G, w) is called a *contra pair* if it has no faithful circuit cover, and a contra pair (G, w) is *minimal* if it has no removable circuit, but for each weight 2 edge e, the graph $G - e$ has a faithful circuit cover with respect to the weight w. It is proved by Alspach et al. (1994) [2] that if (G, w) is a minimal contra pair, then the graph G must *contain a Petersen minor*. It is further conjectured by Fleischner and Jackson (1988) [5] that this graph G must be *the Petersen graph* itself (not just as a minor). In this paper, we prove that this conjecture is true if every Hamilton weight graph is constructed from K_4 via a series of $(Y \to \triangle)$-operations.

1. Introduction

Let G be a graph and let $w : E(G) \mapsto Z^+$ be a weight on the edge set of G. The weighted graph (G, w) is called an *admissible pair* if

(1) the total weight of every edge-cut of G is even,
(2) for every edge-cut T and every $e' \in T$, $w(e') \leqslant \frac{1}{2} \sum_{e \in T} w(e)$.

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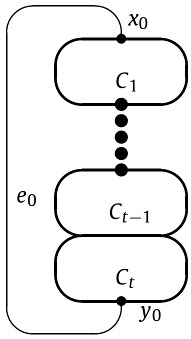


Fig. 1. A circuit chain $C_1, \dots, C_t$ joining the endvertices x_0 and y_0 of e_0.

A weight w satisfying the requirement (1) is called an *eulerian weight*. Obviously, if (G, w) is an admissible pair, then the graph G must be bridgeless. Most graphs considered in this paper are cubic or subgraphs of cubic graphs, and most eulerian weights considered in this paper have the range $\{1, 2\}$.

A family $\mathcal{F}$ of circuits (even subgraphs) is a *faithful circuit (even subgraph) cover* of an admissible pair (G, w) if every edge e of G is contained in precisely $w(e)$ members of $\mathcal{F}$.

A circuit C of G is *removable* in (G, w) if the graph obtained from G by deleting all weight 1 edges contained in C remains bridgeless. An admissible pair (G, w) is a *contra pair* if it has no faithful circuit cover, and a contra pair is *minimal* if G is cubic and (G, w) has no removable circuit, but, for every weight 2 edge e, the graph $G - e$ has a faithful circuit cover with respect to the weight w.

It is proved by Alspach et al. [2] that if (G, w) is a minimal contra pair, then the graph G must *contain a Petersen minor*. It is further conjectured by Fleischner and Jackson ([5], also see [8]) that this graph G must be *the Petersen graph* itself (not just as a minor). In this paper, we prove that this conjecture is implied by Conjecture 1.3.

1.1. Circuit chain

Let G be a graph and $x_0, y_0 \in V(G)$. A family $\{C_1, \dots, C_t\}$ of circuits of G is called a *circuit chain joining x_0 and y_0* if $x_0 \in V(C_1) - \bigcup_{i=2}^{t} V(C_i)$, $y_0 \in V(C_t) - \bigcup_{j=1}^{t-1} V(C_j)$, $V(C_i) \cap V(C_j) \neq \emptyset$ if and only if $i = j \pm 1$ for every $\{i, j\} \subseteq \{1, \dots, t\}$ (see Fig. 1). The circuit chain technique introduced by Seymour [12] is a very powerful and useful method in the study of circuit covers.

In the study of the circuit double cover conjecture [12,13], one often considers a 3-connected cubic graph G that $G - e_0$ has a circuit double cover $\mathcal{C}$ for some edge $e_0 \in G$. One would like to find certain ways to adjust a circuit chain $\{C_1, \dots, C_t\}$ $(\subseteq \mathcal{C})$ joining the endvertices of e_0 so that the missing edge e_0 would be included after adjustment. In order to make such an adjustment, one of very necessary steps is to have some structural results on a circuit chain $\{C_1, \dots, C_t\}$ and, in particular, on the subgraphs in the chain.

In the Workshop on the Cycle Double Cover Conjecture (University of British Columbia, August 22–31, 2007), circuit chain techniques and its structure, as well as related topics such as removable circuits and Hamilton weights, were discussed extensively by participants. This paper reports some results that were stimulated by discussions at the workshop.

1.2. Hamilton weight

As we mentioned above, in order to study the structure of circuit chains and make possible adjustments of circuit covers, one of most basic and natural steps is the characterization of the subgraph induced by two incident circuits. This motivates us to study admissible pairs with precisely two Hamilton circuits as a faithful cover.

Definition 1.1. Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If every faithful circuit cover of the admissible pair (G, w) is a pair of Hamilton circuits, then w is a *Hamilton weight* of G, and (G, w) is called a *Hamilton weight pair*.

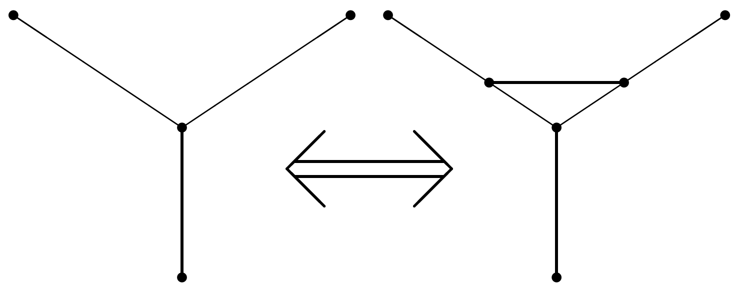


Fig. 2. $(Y - \Delta)$ operation.

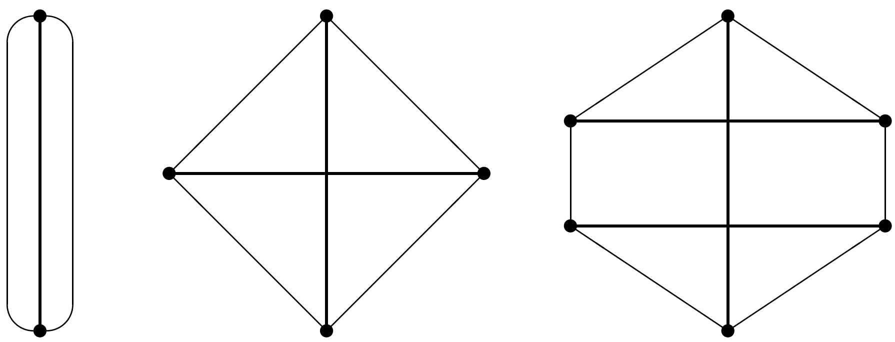


Fig. 3. Three small $\langle\mathcal{K}_4\rangle$-graphs.

Definition 1.2. Let G be a cubic graph and let $v_1v_2v_3v_1$ be a triangle of G. A $(Y \leftarrow \Delta)$-operation of G is to contract the triangle $v_1v_2v_3v_1$ to be a single vertex. A $(Y \rightarrow \Delta)$-operation is the reverse operation of a $(Y \leftarrow \Delta)$-operation (that is, a single vertex is expanded to be a triangle). In general, a $(Y - \Delta)$-operation means either a $(Y \leftarrow \Delta)$- or a $(Y \rightarrow \Delta)$-operation.

Fig. 2 describes the $(Y - \Delta)$-operation. (Note that the assignment of an eulerian weight is also illustrated in the figure: bold lines are weight 2 edges, and slim lines are weight 1 edges.)

Conjecture 1.3. *(See [15].) Let (G, w) be a Hamilton weight pair. If G is 3-connected, then the graph G must be obtained from K_4 by a series of $((Y \rightarrow \triangle))$-operations.*

This conjecture was proved for the family of *Petersen-minor free* graphs [10] and its relation with the problem of *uniquely* 3*-edge-coloring* can be found in [15,16].

Definition 1.4. The family of cubic graphs constructed from $3K_2$ via a series of $((Y \rightarrow \triangle))$-operations is denoted by $\langle\mathcal{K}_4\rangle$.

The three smallest graphs in $\langle\mathcal{K}_4\rangle$ are illustrated in Fig. 3. A Hamilton weight is also illustrated in the figure: bold lines are weight 2 edges, and slim lines are weight 1 edges. The graph $3K_2$ is the graph with two vertices and three parallel edges.

Because of the close relation of circuit double cover and graph embedding, the following two problems are equivalent in some sense:

(1) *the study of the structure of graphs admitting Hamilton weights*;
(2) *for cubic graphs embedded on some surfaces with lowest genus, the study of the local structure of two incident faces.*

There is a close relationship between the study of a cubic graphs admitting Hamilton weights and embeddings of cubic graphs on a surface of minimal genus, see Zha [14] and Sanders [11].

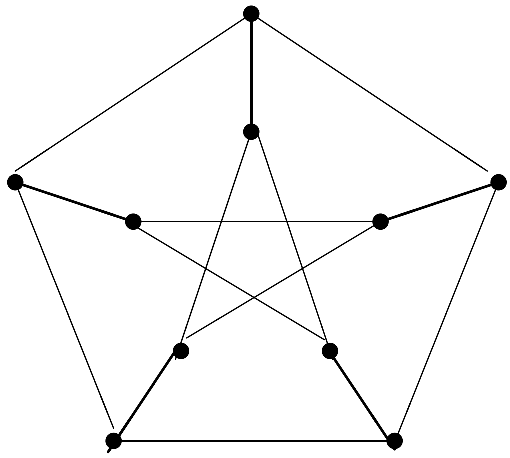


Fig. 4. The Petersen graph P_{10} and the weight w_{10}.

1.3. Fleischner–Jackson conjecture and other major conjectures

It is proved by Alspach et al. [2] (see Lemmas 4.1) that if (G, w) is a minimal contra pair, then the graph G must be a *permutation graph*. By applying a theorem of Ellingham [4], this graph must *contain a Petersen minor*. Fleischner and Jackson further conjectured that it must be *the Petersen graph itself*.

Conjecture 1.5. *(See Fleischner and Jackson, Conjecture 12 in [5] or see [8].) Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If (G, w) is a minimal contra pair, then $(G, w) = (P_{10}, w_{10})$ where P_{10} is* Petersen graph *and w_{10} is an eulerian weight such that the set of weight 2 edges induces a perfect matching (see Fig. 4).*

Beyond the Petersen-minor free graphs, a series of similar major conjectures related to contra pair and removable circuits have been proposed, each of which suggests that the *Petersen graph* is the only contra pair under various hypothesis.

Conjecture 1.6. *(See Goddyn [7], or see [8].) Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If (G, w) is a contra pair such that G is 3-connected and essentially 4-edge-connected and (G, w) has no removable circuit, then G must be the* Petersen graph.

Conjecture 1.7. *(See Jackson [9].) Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If (G, w) is a contra pair such that G is cyclically 5-edge-connected, then G must be the* Petersen graph.

The following conjecture is more specific than Conjecture 1.7.

Conjecture 1.8. *(See Fleischner, Genest and Jackson [6].) Let G be a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If (G, w) is a contra pair such that, for every cyclic edge-cut T, $\sum_{e \in T} w(e) > 4$, then G must be the* Petersen graph.

It should be noticed [6] that Conjecture 1.8 implies the *circuit double cover conjecture* [12,13].

1.4. Main theorem

We will show that Conjecture 1.3 implies Conjecture 1.5.

Theorem 1.9. *If every 3-connected Hamilton weight cubic graph is a member of $\langle \mathcal{K}_4 \rangle$, then every minimal contra pair must be the* Petersen graph P_{10} *associated by the eulerian weight w_{10}.*

Fig. 4 is the Petersen graph P_{10} associated with the weight w_{10} where bold lines are weight 2 edges, and slim lines are weight 1 edges.

2. Notation and terminology

For notations not defined here see [3] or [16].

Let A and B be two sets. The *symmetric difference* of A and B, denoted by $A\Delta B$, is defined as follows:

$$A\Delta B = (A \cup B) - (A \cap B).$$

Most graphs considered in main theorems, conjectures and lemmas of this paper are cubic. Some subgraphs appearing in the proofs of some theorems or lemmas may have smaller degrees, but their maximum degrees are at most 3.

A *circuit* is a connected 2-regular graph, while an *even subgraph* (or cycle) is a graph with even degree for every vertex. An edge e is a *bridge* of a graph G if the removal of e increases the number of components.

Let $G = (V, E)$ be a graph. The *suppressed graph*, denoted by $\overline{G}$, is the graph obtained from G by suppressing all degree-2-vertices.

An edge-cut T of G is *trivial* if some component of $G - T$ is a single vertex. A graph G is *essentially λ-edge-connected* if every edge-cut T with $|T| < \lambda$ is trivial.

An edge-cut T is *cyclic* if every component of $G - T$ contains some circuit. A graph G is *cyclically λ-edge-connected* if every edge-cut T with $|T| < \lambda$ is not cyclic.

Let w be an eulerian weight of G. The set of edges with weight i is denoted by $E_{w=i}$.

Definition 2.1. Let $\mathcal{C} = \{C_1, \ldots, C_s\}$ be a set of circuits of a graph G. An eulerian weight $w_{\mathcal{C}}$ of G induced by the coverage of $\mathcal{C}$ is defined as follows:

$$w_{\mathcal{C}}(e) = \bigl|\{C \in \mathcal{C}\colon\ e \in C\}\bigr|.$$

It is obvious that $w_{\mathcal{C}}$ is eulerian since $\mathcal{C}$ is a set of circuits.

Definition 2.2. Let (G, w) be an admissible pair. A w-decomposition of (G, w) is a pair $\{(H_1, w_1), (H_2, w_2)\}$ where H_1 and H_2 are subgraphs of G with $H_1 \cup H_2 = G$ and w_i is an eulerian weight of H_i $(i = 1, 2)$ such that

$$w(e) = \begin{cases} w_1(e) & \text{if } e \in H_1 - H_2, \\ w_2(e) & \text{if } e \in H_2 - H_1, \\ w_1(e) + w_2(e) & \text{if } e \in H_1 \cap H_2. \end{cases}$$

Definition 2.3. Let (G, w) be an admissible pair and let $\{(H_1, w_1), (H_2, w_2)\}$ be a w-decomposition of (G, w) such that H_1 is a circuit with $w_1 \equiv 1$. If H_2 is bridgeless, then H_1 is called a *removable circuit* of (G, w).

Definition 2.4. Let G be a cubic graph and let H_1, H_2 be subgraphs of G. An *attachment of H_2 in the suppressed graph $\overline{H_1}$* is an edge $e = uv$ of $\overline{H_1}$ such that the edge e corresponds to a maximal induced path $P = u \cdots v$ (in H_1) and $V(H_2) \cap [V(P) - \{u, v\}] \neq \emptyset$.

Let (G, w) be an admissible pair and $e_0 \in E_{w=2}$ that $G - e_0$ is bridgeless. With no confusion and a slight abuse of notation, the admissible pair with the graph $G - e_0$ and the eulerian weight that preserves the values of w on the edge set $E(G) - \{e_0\}$ is denoted by $(G - e_0, w)$. Similarly, the admissible pair with the graph $\overline{G - e_0}$ and the eulerian weight that preserves the values of w on the edge set $E(\overline{G - e_0})$ is also denoted by $(\overline{G - e_0}, w)$.

3. Lemmas and preliminaries

3.1. 3-edge-colorings and faithful covers

Lemma 3.1. *(See Seymour [12].) Let G be a cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$. If G has a 3-edge-coloring $c : E(G) \mapsto Z_3$, then G has a faithful even subgraph cover consisting of the following three even subgraphs*

$$E_{w=1} \triangle \left[c^{-1}(0) \cup c^{-1}(1)\right], \qquad E_{w=1} \triangle \left[c^{-1}(0) \cup c^{-1}(2)\right], \qquad E_{w=1} \triangle \left[c^{-1}(1) \cup c^{-1}(2)\right].$$

Definition 3.2. Let (G, w) be an admissible pair. Suppose that c is a 3-edge-coloring of the cubic graph G. The faithful circuit cover of (G, w) described in Lemma 3.1 is called *a faithful cover induced by the coloring c*.

3.2. Structural lemmas about Hamilton weights

Lemmas 3.3 and 3.4 are rather straightforward observations. They are used as preparations for the proof of a major lemma (Lemma 3.5) in this subsection.

Lemma 3.3. *Let (G, w) be a Hamilton weight pair. Then the total weight of every edge cut of G is at least 4.*

The following lemma is an immediate corollary of Lemma 3.3.

Lemma 3.4. *Let (G, w) be a Hamilton weight pair. If $E_{w=1}$ induces a Hamilton circuit of G, then G is 3-connected.*

Lemma 3.5. *Let G be a bridgeless, cubic graph and let $w : E(G) \mapsto \{1, 2\}$ be an admissible eulerian weight of G. Let $e_0 \in E_{w=2}$. Suppose that*

(1) *$E_{w=1}$ induces a Hamilton circuit of G,*
(2) *every removable circuit of (G, w) must contain the edge e_0.*

Then (G, w) is a Hamilton pair, and, with the assumption that Conjecture 1.3 is true, the graph $G \in \langle \mathcal{K}_4 \rangle$.

Proof. Since $E_{w=1}$ induces a Hamilton circuit, the cubic graph G is 3-edge-colorable. Hence, by Lemma 3.1, the admissible pair (G, w) has a faithful circuit cover $\mathcal{C}$. Since every member of $\mathcal{C}$ not containing e_0 is a removable circuit of (G, w) avoiding e_0, the weight 2 edge e_0 is contained in *every member* of $\mathcal{C}$. That is, $|\mathcal{C}| = 2$, and, therefore, (G, w) is a Hamilton weight pair. By Lemma 3.4, G is 3-connected. The lemma is proved under the assumption of Conjecture 1.3. $\square$

3.3. Structural lemmas for $\langle \mathcal{K}_4 \rangle$-graphs

In Lemma 3.6, we deal with some special edges e of a $\langle \mathcal{K}_4 \rangle$-graph G with the property that *the suppressed cubic graph $\overline{G - e}$ remains in $\langle \mathcal{K}_4 \rangle$*. It is obvious that every edge contained in a triangle of a $\langle \mathcal{K}_4 \rangle$-graph G has this property. Actually edges with this property may not necessary be contained in triangles. (This type of edges will be further discussed in this subsection and in Section 6.4.)

Most lemmas in this subsection are very technical. Readers are suggested to read the lemmas and their proofs whenever they are needed in the proof of Theorem 1.9. Proofs of some lemmas will be presented in Section 6.

Lemma 3.6 will be applied in the proof of Lemma 4.4 for the structures of a circuit chain.

Lemma 3.6. *Let $G \in \langle \mathcal{K}_4 \rangle$ of order at least 4, and $e = v_1 v_2 \in E(G)$. For each $i = 1, 2$, let e_i be the edge of $\overline{G - e}$ containing the vertex v_i (that is, e_i is an attachment of e in the suppressed graph $\overline{G - e}$). Assume that*

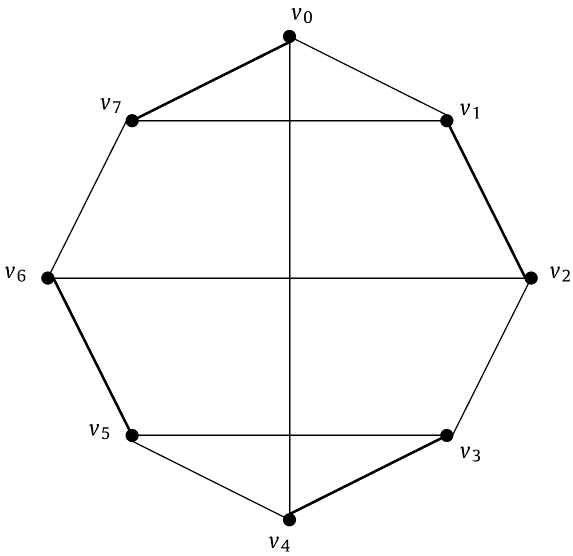


Fig. 5. An L-graph.

every 3-edge-cut of G containing e is trivial (*that is*, $E(v_1)$ *and* $E(v_2)$ *are the only two* 3-*edge-cuts of* G *containing the edge* e). *Then we have the following conclusions.*

(1) *The suppressed cubic graph* $\overline{G-e} \in \langle \mathcal{K}_4 \rangle$;
(2) $\{e_1, e_2\}$ *is contained in a* 3-*edge-cut of* $\overline{G-e}$;
(3) *If* w *is a Hamilton weight of* G *and* $w(e) = 2$, *then* $(\overline{G-e}, w)$ *remains as a Hamilton weight pair.*

Proof. See Section 6.2. □

Definition 3.7. Let $G \in \langle \mathcal{K}_4 \rangle$ of order $2n$ ($\geqslant 4$). The graph G is called an L-graph if G has a Hamilton circuit

$$C_2 = v_0, \dots, v_{2n-1}v_0,$$

with a *diagonal crossing chord* v_0v_n and a set Z of *parallel chords* where

$$Z = \{v_{2n-\mu}v_\mu\colon \mu = 1, \dots, n-1\}.$$

An admissible pair (G, w) is called an *an L-graph pair* if G is an L-graph and w is a Hamilton weight with

$$E_{w=2} = \{v_{2i-1}v_{2i}\colon i = 1, \dots, n\}.$$

Fig. 5 is an illustration of an L-graph pair with 8 vertices. Note that, in Fig. 5, bold lines are edges in $E_{w=2}$ and slim lines are edges in $E_{w=1}$.

Draw an L-graph on the plane such that the Hamilton circuit C_2 is the boundary of the exterior region and all chords are in the interior region bounded by C_2. Then one can see that all parallel chords do not cross each other, while the diagonal crossing chord crosses every parallel chord. We notice that an L-graph can be constructed recursively from $3K_2$ as follows: for a given edge e^* of $3K_2$ (if a Hamilton weight is assigned, choose $e^* \in E_{w=1}$), an L-graph can be obtained from $3K_2$ via a series of $(Y \to \Delta)$-operations *only* at an endvertex of e^*. (Note that the edge e^* will remain as the diagonal crossing chord during the expansion of the L-graph.)

Lemma 3.8. *Let* $G \in \langle \mathcal{K}_4 \rangle$ *of order* $2n$ ($\geqslant 4$) *and associated with a Hamilton weight* w. *Let* $\{C_1, C_2\}$ *be a faithful circuit cover of* (G, w). *Let* $e \in C_1 - C_2$ *and* $F \subseteq C_2 - C_1$. *Assume that*

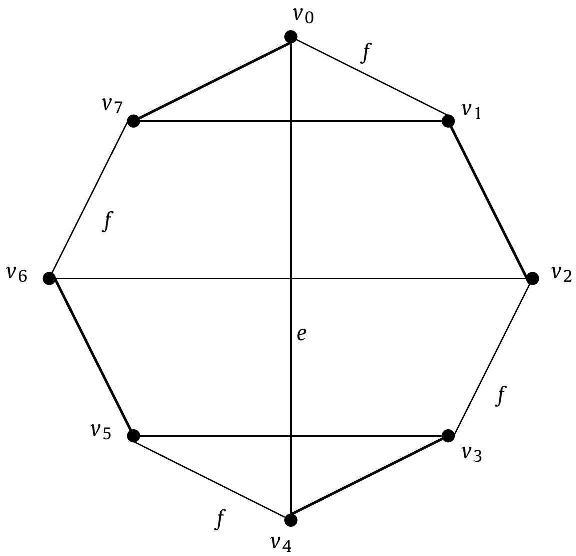


Fig. 6. An L-graph with specified edges e and F.

(a) *every triangle of G contains some edge of $F \cup \{e\}$, and*
(b) *for every edge $f \in F$, G contains a 3-edge-cut T with both $f, e \in T$.*

Then (G, w) must be an L-graph pair as described in Definition 3.7 with $e = v_0 v_n$ (as the diagonal crossing chord) and

$$\{v_0 v_1, v_\nu v_{\nu+1}\} \subseteq F \subseteq \{v_{2i} v_{2i+1}\colon\ i = 0, \ldots, n-1\} = C_2 - C_1$$

where $\nu = n$ if n is even and $\nu = n + 1$ if n is odd.

Proof. See Section 6.3. □

Fig. 6 is an illustration of an L-graph pair with 8 vertices. Note that, in Fig. 6, bold lines are edges in $E_{w=2}$ and slim lines are edges in $E_{w=1}$; the circuit $C_2 = v_0 \cdots v_7 v_0$; the circuit $C_1 = v_0 v_7 v_1 v_2 v_6 v_5 v_3 v_4 v_0$; edges labeled with f are possible locations of edges of F.

4. Lemmas for minimal contra pair

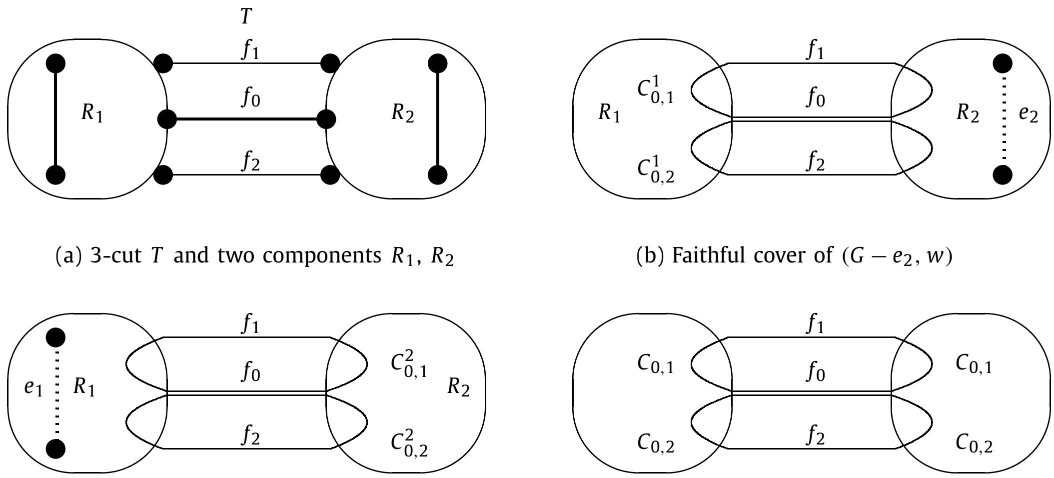
Let (G, w) be a minimal contra pair. That is, G is a bridgeless cubic graph associated with an eulerian weight $w : E(G) \mapsto \{1, 2\}$ and

(1) *(G, w) has no faithful circuit cover,*
(2) *for every weight 2 edge e, $(G - e, w)$ has a faithful circuit cover, and*
(3) *(G, w) has no removable circuit.*

Before the final proof of Theorem 1.9, a list of lemmas are presented in this section.

Lemma 4.1. *(See Alspach, Goddyn and Zhang [2].) Let (G, w) be a minimal contra pair. Then*

(1) *for each weight 2 edge $e = xy$ of G, every faithful circuit cover of $(G - e, w)$ is a circuit chain joining the vertices x and y;*
(2) *the subgraph of G induced by weight 1 edges is a 2-factor of G consisting of two chordless circuits Q_1 and Q_2 each of which is of odd length, and the subgraph of G induced by weight 2 edges is a perfect matching M joining Q_1 and Q_2.*



(a) 3-cut T and two components R_1, R_2

(b) Faithful cover of $(G - e_2, w)$

(c) Faithful cover of $(G - e_1, w)$

(d) Faithful cover $\mathcal{C}^1 \cup \mathcal{C}^2$ of (G, w)

Fig. 7. Small cut T in minimal contra pair (G, w).

The proof of Lemma 4.1 can be found in [1,2] (and also see [16]).

Note that these two properties will be used *frequently* in the remaining part of the proof. In this and next sections, we will standardize the notations as follows:

For a minimal contra pair (G, w),
(1) *let* Q_1 *and* Q_2 *be the pair of chordless circuits induced by edges of* $E_{w=1}$, *and*
(2) $M = E_{w=2}$ *be the perfect matching joining* Q_1 *and* Q_2.

Lemma 4.2. *Let* (G, w) *be a minimal contra pair, let* $e_0 = x_0 y_0$ *be an arbitrary weight* 2 *edge of* (G, w), *and let* $\mathcal{C} = \{C_1, \ldots, C_t\}$ *be a faithful cover of* $(G - e_0, w)$ *with* t *as large as possible. Then* $t \geqslant 3$.

Proof. Recall Lemma 4.1(2), Q_1 and Q_2 are the pair of chordless circuits of odd length and M is the perfect matching joining Q_1 and Q_2.

Suppose that $\mathcal{C} = \{C_1, C_2\}$ $(t = 2)$. In the suppressed cubic graph $\overline{G - e_0}$, alternatively color the edges of Q_j $(j = 1, 2)$ with red and blue such that the subdivided edges containing x_0 or y_0 are both colored with red. Let M_1 be the set of all red edges and let M_2 be the set of all blue edges in G. Then $\{M_1 \cup M, M_2 \cup M\}$ is a faithful even subgraph cover of $(G - e_0, w)$. But, both x_0 and y_0 are contained in the even subgraph $M_1 \cup M$, which is a circuit because of the maximality of $|\mathcal{C}|$, and so $M_2 \cup M$ is removable. This is a contradiction. □

Lemma 4.3. *Let* (G, w) *be a minimal contra pair. Then* G *is essentially* 4*-edge-connected.*

Proof. Suppose that T is a *non-trivial edge-cut* of G separating G into two components R_1 and R_2 with

$$2 \leqslant |T| \leqslant 3. \tag{1}$$

I. We claim that *either* $R_1 \cap E_{w=2} = \emptyset$ *or* $R_2 \cap E_{w=2} = \emptyset$.

Suppose that $R_1 \cap E_{w=2} \neq \emptyset$ and $R_2 \cap E_{w=2} \neq \emptyset$ (see Fig. 7(a)). Choose an edge $e_i \in E_{w=2} \cap R_i$ $(i = 1, 2)$. By the definition of *minimal contra pair* that $(G - e, w)$ has a faithful circuit cover for *every* weight 2 edge e (see Section 1), let $\mathcal{C}^i_0$ be a faithful cover of $(G - e_{3-i}, w)$ (see Fig. 7(b) and (c)). For each $i = 1, 2$, since the un-covered edge e_{3-i} is not in $T \cup R_i$, every edge of $T \cup R_i$ is *faithfully covered* by $\mathcal{C}^i_0$. Thus, we further let $\mathcal{C}^i$ be a faithful circuit cover of $(G/R_{3-i}, w)$, for each i, where $\mathcal{C}^i$ is obtained from $\mathcal{C}^i_0$ by deleting members that are completely contained in R_{3-i}, and contracting edges of R_{3-i} in each remaining member of $\mathcal{C}^i_0$.

Consider the case that $|T| = 3$. Since w is a $(1,2)$-eulerian weight and T is a 3-edge-cut of G, the cut T contains either one or three weight 2 edges. Thus, the total weight of T is $2h$ where $h =$ either 2 or 3, and, therefore, there are h *distinct* members of $\mathcal{C}^i$ passing through the cut T. Let $T = \{f_1, f_2, f_3\}$, and let $C^i_{\alpha,\beta}$ be the member of $\mathcal{C}^i$ containing f_α, f_β $(\in T)$ (see Fig. 7(b) and (c)). In $\mathcal{C}^1 \cup \mathcal{C}^2$, for each $\{\alpha, \beta\} \subset \{1,2,3\}$, replacing $\{C^1_{\alpha,\beta}, C^2_{\alpha,\beta}\}$ with a single circuit $C_{\alpha,\beta}$ such that $E(C_{\alpha,\beta}) \cap (E(R_i) \cup T) = E(C^i_{\alpha,\beta}) \cap (E(R_i) \cup T)$ (see Fig. 7(d)), we obtain a faithful circuit cover of (G, w).

It is similar for the case of $|T| = 2$. This contradicts that (G, w) is a contra pair.

II. By I, without loss of generality, let

$$R_2 \cap E_{w=2} = \emptyset.$$

Since G is cubic and $M = E_{w=2}$ is a perfect matching (by Lemma 4.1), every vertex of G must be incident with one edge of $E_{w=2}$. Since $E_{w=2} \cap R_2 = \emptyset$, for every vertex $v \in R_2$, the edge of $E_{w=2} \cap E(v)$ must be contained in T (not in $E(R_2)$). Hence,

$$\left|V(R_2)\right| = |E_{w=2} \cap T|. \tag{2}$$

Note that $E_{w=1}$ induces a 2-factor of the graph G, we must have that $|T \cap E_{w=1}| \equiv 0 \pmod 2$.

If $|T \cap E_{w=1}| = 2$, then $|E_{w=2} \cap T| = |T| - 2$. By inequality (1) and equations (2), we have that $|V(R_2)| \leqslant 1$. This contradicts the assumption that T is a non-trivial edge-cut.

If $|T \cap E_{w=1}| = 0$, then $|E_{w=2} \cap T| = |T|$. By inequality (1) and equation (2), we have that $|V(R_2)| = 2$ or 3. Thus, R_2 is a digon or a triangle and is a component (say, Q_2) of the 2-factor $E_{w=1}$. Hence, by Lemma 4.1(2), we have that $T = E_{w=2}$ and R_1 is another component Q_1 of $E_{w=1}$. Now, one can easily see that the graph G has only 4 or 6 vertices and it must be 3-edge-colorable. By Lemma 3.1, this contradicts that (G, w) is a contra pair. □

The next lemma (Lemma 4.4) is one of the most complicated parts of the proof for Theorem 1.9. In order to provide an easy reading for readers, we will present some special notations and outline some major claims before the detailed proof.

Notation 4.1. Let $\{C_1, \ldots, C_t\}$ be a circuit chain of $G - e_0$ joining the end vertices of e_0 with $t \geqslant 3$. For $1 \leqslant i < j \leqslant t$, let $H_{i,j} = \overline{G[\bigcup_{\mu=i}^{j} E(C_\mu)]}$ be the suppressed graph of the subgraph of G induced by all edges of $C_i, \ldots, C_j$, and let $w_{i,j}$ be the eulerian weight of $H_{i,j}$ induced by the coverage of $\{C_i, \ldots, C_j\}$ (see Definition 2.1 and Fig. 8).

Lemma 4.4. *Let (G, w) be a minimal contra pair, let $e_0 = x_0 y_0$ be a weight 2 edge of (G, w), and let $\mathcal{C} = \{C_1, \ldots, C_t\}$ be a faithful cover of $(G - e_0, w)$ with t as large as possible. If the Hamilton weight conjecture is true, then*

(1) $t = 3$,
(2) $(H_{1,2}, w_{1,2})$ *(and $(H_{2,3}, w_{2,3})$) is an L-graph pair in which the diagonal crossing chord is an attachment of e_0.*

In the proof of this lemma, the assumption that Conjecture 1.3 is true is only used indirectly, when we apply Lemma 3.8 in the proof of Claim 4.2.

Proof.

A. Outlines and notations. Since the length of the circuit chain is maximized, by Lemma 4.2, $t \geqslant 3$.

Claim 4.1. *$E_{w_{1,t-1}=1}$ induces a Hamilton circuit of $H_{1,t-1}$.*

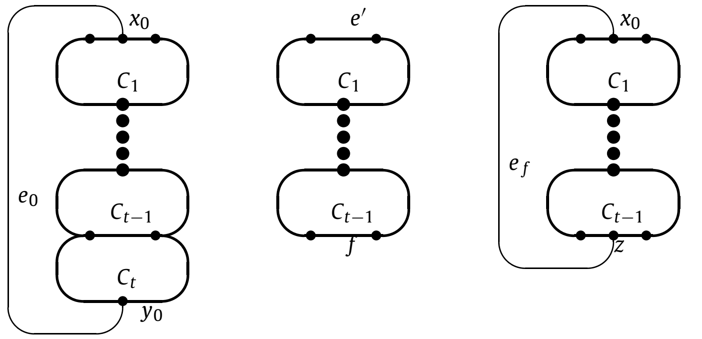


Fig. 8. Circuit chain and subchain.

Notation 4.2. Let e' be the attachment of e_0 in $H_{1,t-1}$, and F be the set of all attachments of C_t in $H_{1,t-1}$. That is, e' corresponds to an induced path in the subgraph $G[\bigcup_{i=1}^{t-1} E(C_i)]$ containing the vertex x_0, and, for each $f \in F$, f corresponds to an induced path in $G[\bigcup_{i=1}^{t-1} E(C_i)]$ containing some vertex of C_t. (See Fig. 8.)

Notation 4.3. For each $f \in F$, construct a new graph H_f from $H_{1,t-1}$ as follows. Insert x_0 back into the edge e' and insert a new vertex z into the edge f, and add a new weight 2 edge e_f joining x_0 and z. Let w_{H_f} be the resulting eulerian weight of H_f. (See Fig. 8.)

In order to apply Lemma 3.6 in the proof of Claim 4.6, we will first prove the following claims.

Claim 4.2. *(H_f, w_{H_f}) is a Hamilton weight pair and $H_f \in \langle \mathcal{K}_4 \rangle$.*

Claim 4.3. *$|F| \geqslant 2$ and $|V(H_{1,t-1})| \geqslant 4$.*

Claim 4.4. *Every 3-edge-cut of H_f containing e_f is trivial.*

Claim 4.5. *$t = 3$.*

Claim 4.6. *$H_{1,2}, H_{2,3} \in \langle \mathcal{K}_4 \rangle$; $(H_{1,2}, w_{1,2})$ and $(H_{2,3}, w_{2,3})$ are Hamilton weight pairs.*

The final conclusion is the following claim which is proved by applying Lemma 3.8.

Claim 4.7. *$(H_{1,2}, w_{1,2})$ (and $(H_{2,3}, w_{2,3})$) is an L-graph pair in which the diagonal crossing chord is the attachment of e_0 (the edge e' of $H_{1,2}$ defined in Notation 4.2).*

B. Proof of Claim 4.1. Let X be the subgraph of $H_{1,t-1}$ induced by $E_{w_{1,t-1}=1}$. Suppose that $\{X_1, \ldots, X_r\}$ is the set of components of X with $r \geqslant 2$.

We define a 3-edge-coloring c of $H_{1,t-1}$ as follows:

$$c(e) = \begin{cases} \text{red} & \text{if } e \in [\bigcup_{j=1}^{\lceil \frac{t}{2} \rceil} E(C_{2j-1})] - [\bigcup_{j=1}^{\lfloor \frac{t}{2} \rfloor} E(C_{2j})], \\ \text{blue} & \text{if } e \in [\bigcup_{j=1}^{\lfloor \frac{t}{2} \rfloor} E(C_{2j})] - [\bigcup_{j=1}^{\lceil \frac{t}{2} \rceil} E(C_{2j-1})], \\ \text{purple} & \text{if } e \in [\bigcup_{j=1}^{\lceil \frac{t}{2} \rceil} E(C_{2j-1})] \cap [\bigcup_{j=1}^{\lfloor \frac{t}{2} \rfloor} E(C_{2j})]. \end{cases}$$

Note that

(1) *all attachments of C_t in $H_{1,t-1}$ are colored with the same color, since $\mathcal{C}$ is a circuit chain (see Lemma 4.1(1)).*
(2) *edges of each component of X are alternately colored with red and blue.*

We claim that *the circuit C_t intersects with only one component of X*. Suppose that the attachments of C_t in $H_{1,t-1}$ are contained in at least two components $X_1, X_2, \ldots$. By interchanging the colors around the circuit X_2, one obtains a different 3-edge-coloring c' of the graph $H_{1,t-1}$. Let $\mathcal{D} = \{D_1, \ldots, D_s\}$ be a faithful circuit cover of $(H_{1,t-1}, w_{1,t-1})$ induced by the coloring c' as follows: $\bigcup_{j=1}^{a} E(D_j)$ consists of all red–purple edges, and $\bigcup_{j=a+1}^{s} E(D_j)$ consists of all blue–purple edges, for some integer a: $1 \leqslant a < s$. Note that $\{D_1, \ldots, D_s, C_t\}$ is another faithful circuit cover of $(G - e_0, w)$ with $x_0 \in D_i$ (for some $i \in \{1, \ldots, s\}$) and $y_0 \in C_t$. We further notice that the circuit C_t intersects with two distinct members of $\{D_1, \ldots, D_s\}$, one of which is red–purple colored, while another one is blue–purple colored. This contradicts Lemma 4.1(1) that every faithful cover of $(G - e_0, w)$ must be a circuit chain.

So, we must have that C_t intersects with only one component of X, say X_1. Since C_t does not intersect with X_2, the circuit X_2, which consists of some edges of $E_{w=1}$, must be one of Q_1 and Q_2 – the components of the 2-factor $G[E_{w=1}]$ of the permutation graph (see Lemma 4.1(2)). But, one can easily see that every weight 2 edge in $C_t \cap X_1$ cannot be an edge of the perfect matching M joining Q_1 and Q_2. This contradicts Lemma 4.1(2) that G is a permutation graph.

C. Proof of Claim 4.2. Since $E_{w_{1,t-1}=1} = E_{w_{H_f}=1}$ induces a Hamilton circuit X in $H_{1,t-1}$ (by Claim 4.1), it remains as a Hamilton circuit of H_f. Since any removable circuit of (H_f, w_{H_f}) avoiding the edge e_f is also a removable circuit of (G, w), by Lemma 3.5, (H_f, w_{H_f}) is a Hamilton weight pair and $H_f \in \langle \mathcal{K}_4 \rangle$.

D. Proof of Claim 4.3. Suppose that $|F| = 1$. Then we can find a non-trivial 3-edge-cut of G consisting of two edges of C_{t-1} (part of the induced path corresponding to f) and e_0. Note that this edge-cut is not trivial since $t \geqslant 3$. This contradicts Lemma 4.3 that G is essentially 4-edge-connected.

Suppose that $|V(H_{1,t-1})| = 2$. Then $H_{1,t-1} = 3K_2$ and therefore, $|F| = 1$. This contradicts the above result that $|F| \geqslant 2$.

E. Proof of Claim 4.4. Suppose that T is a non-trivial 3-edge-cut of H_f containing the edge e_f.

I. Since (by Claim 4.1) $E_{w_{H_f}=1}$ induces a Hamilton circuit of H_f, we have that $T - e_f$ is a 2-edge-cut of $H_{1,t-1}$ consisting of only $E_{w=1}$-edges. So, there is an integer k: $1 \leqslant k \leqslant t-1$, such that

$$T - e_f \subseteq C_k \cap E_{w=1} \tag{3}$$

since $H_{1,t-1}$ is induced by the circuit chain $C_1, \ldots, C_{t-1}$.

Let X', X'' be two components of $H_f - T$ such that $x_0 \in X'$ and $z \in X''$. (Recall that $e_f = x_0 z$ where x_0 is the endvertex of e_0 contained in C_1 and z is the vertex inserted into the edge f of C_{t-1} – see Notation 4.3.)

By Eq. (3),

$$X' \subseteq C_1 \cup \cdots \cup C_k, \qquad X'' \subseteq C_k \cup \cdots \cup C_{t-1}. \tag{4}$$

II. By Eq. (3), each $E_{w=2}$-edge of $H_{1,t-1}$ incident with some vertex of X'' cannot be in $T - e_f$ and therefore (by Eq. (4)), must be in the intersection of $C_i \cap C_{i+1}$ for some $i \geqslant k$, and is contained in X''. Since T is a non-trivial cut, X'' contains some $E_{w=2}$-edges and therefore

$$k < t - 1. \tag{5}$$

III. Recall that F is the set of all attachments of C_t in $H_{1,t-1}$. We further notice that if $F \cap X' = \emptyset$, then $T - e_f + e_0$ is a non-trivial 3-edge-cut of G. This contradicts Lemma 4.3 that G is essentially 4-edge-connected. So, we must have that $F \cap X' \neq \emptyset$.

Let $f' \in F \cap X'$. Since f' is an attachment of C_t in $H_{1,t-1}$ and is contained in C_{t-1}, we have that $C_{t-1} \cap X' \neq \emptyset$, and therefore, by (4), $k \geqslant t - 1$. This contradicts inequality (5).

F. Proof of Claims 4.5 and 4.6. Lemma 3.6 is ready to be applied here. By Claim 4.2, $H_f \in \langle \mathcal{K}_4 \rangle$ and by Claim 4.4, every 3-edge-cut of H_f containing e_f is trivial. Therefore, by Lemma 3.6(1) and (3), $H_{1,t-1} = \overline{H_f - e_f} \in \langle \mathcal{K}_4 \rangle$, $w_{1,t-1}$ is a Hamilton weight of $H_{1,t-1}$ and therefore, $t - 1 = 2$.

Symmetrically, $H_{2,t} = H_{2,3}$ has the similar properties as that of $H_{1,2}$.

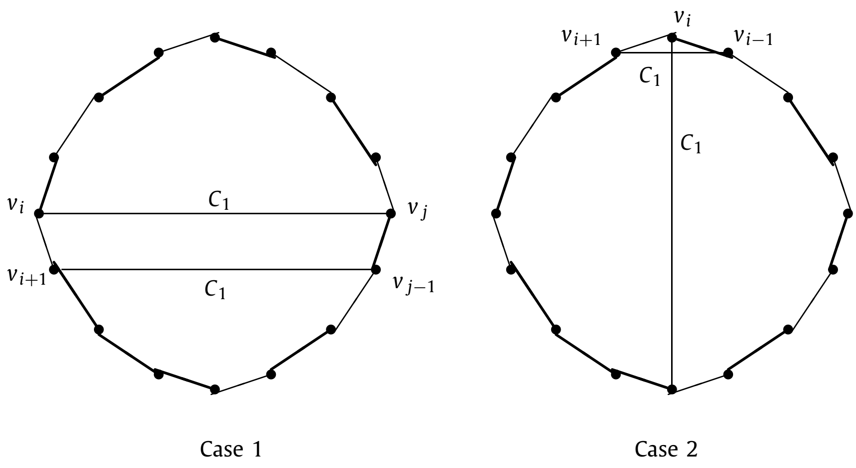


Fig. 9. $E_{w=2}$-edges around the circuit C_2.

G. Proof of Claim 4.7. Continue from F, by Lemma 3.6(2), we have that, *for each* $f \in F$, *both* e' *and* f *are contained in some* 3*-edge-cut of* $H_{1,2}$ (hypothesis (b) of Lemma 3.8). Since G is essentially 4-edge-connected (by Lemma 4.3), *every triangle of* $H_{1,2}$ *must contain some edge of* $F \cup \{e'\}$, for otherwise, the set of pending edges of a triangle is a non-trivial 3-edge-cut of G (hypothesis (a) of Lemma 3.8).

By Lemma 3.8, $(H_{1,2}, w_{1,2})$ must be an L-graph pair with e' (an attachment of e_0) as the diagonal crossing chord.

Similarly, the graph $H_{2,3} = \overline{C_2 \cup C_3}$ is also an L-graph with an attachment of e_0 as the diagonal crossing chord. $\square$

5. Proof of the main theorem (Theorem 1.9)

We continue to use the notations defined in Section 4.

Let $e_0 = x_0y_0$ be an arbitrary edge of G with $w(e_0) = 2$, and let $\mathcal{C} = \{C_1, C_2, C_3\}$ be a faithful circuit cover of $(G - e_0, w)$ (Lemma 4.4).

By Lemma 4.4, C_2 is a Hamilton circuit of the suppressed graph $\overline{G - e_0}$. It is obvious that edges of C_2 are alternately in $E_{w=1}$ and $E_{w=2}$ since $C_1 \cap C_3 = \emptyset$.

Claim 5.1. *Around the circuit* C_2, $E_{w=2}$*-edges are alternately in* C_1 *and* C_3.

Assume that there is an $E_{w=1}$-edge v_iv_{i+1} ($\in E(C_2)$) such that both edges $v_{i-1}v_i$ and $v_{i+1}v_{i+2} \in C_1 \cap C_2$. (See Fig. 9.)

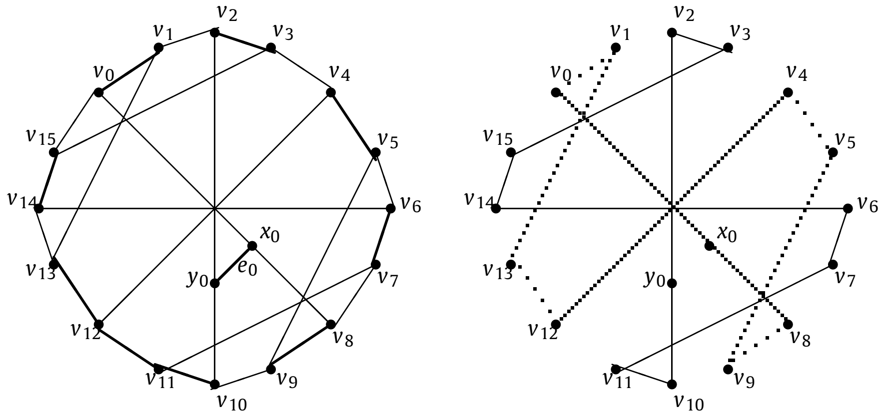
Case 1. *Neither* v_i *nor* v_{i+1} *is incident with the diagonal crossing chord of the* L*-graph* $\overline{G[C_1 \cup C_2]}$. Then, by the structure of an L-graph (see Definition 3.7), $D = v_iv_jv_{j-1}v_{i+1}v_i$ is a 4-circuit of G where $v_jv_{j-1} \in C_1 \cap C_2$, v_iv_j and $v_{j-1}v_{i+1}$ are a pair of parallel chords. Note that v_jv_{j-1} is the only $E_{w=2}$-edge in this 4-circuit D. This contradicts Lemma 4.1(2) that G is a permutation graph in which each component of $E_{w=1}$ is chordless.

Case 2. *The vertex* v_i *is incident with the diagonal crossing chord of the* L*-graph* $\overline{G[C_1 \cup C_2]}$. In this case (see Definition 3.7), $v_{i-1}v_iv_{i+1}v_{i-1}$ is a triangle of the graph G. As in Case 1, this does not happen since G is a permutation graph.

So, by Claim 5.1, the number of $E_{w=2}$-edges in C_2 is even, and therefore,

$$\left|E(C_2)\right| \equiv 0 \pmod 4. \tag{6}$$

(See Fig. 10 for an illustration of these circuits in G with $|V(G)| = 18$. Note that, in the first graph of Fig. 10, bold lines are edges in $E_{w=2}$ and slim lines are edges in $E_{w=1}$.)

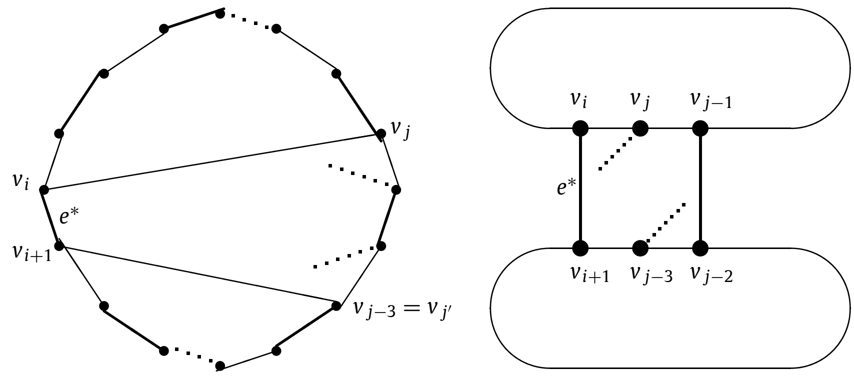


The circuit C_2: $v_0 \cdots v_{15} v_0$

Solid lines: the circuit C_3

Dash lines: the circuit C_1

Fig. 10. The graph G with $|V(G)| = 18$.



(α) A 6-circuit (C_2 and its chords) (β) A 6-circuit (in permutation graph)

Fig. 11. A 6-circuit containing an edge of $E_{w=2}$.

Claim 5.2. $|V(G)| = 10$.

After this claim (Claim 5.2) is proved, we will be able to show that the graph G with 10 vertices must be the Petersen graph in the next subsection.

By Eq. (6), $|C_2| = 4h$, for some integer h. By Claim 5.1, the L-graph $\overline{G[C_1 \cup C_2]}$ is of order $2h$. By the definition of L-graph (Definition 3.7), $h \geqslant 2$.

We are to find some special circuits containing edges $e \in M = E_{w=2}$. Since the edge $e_0 \in M = E_{w=2}$ was *arbitrarily* selected, every edge $e \in M = E_{w=2}$ must have similar properties as that of $e_0 = x_0 y_0$.

Assume that $h \geqslant 3$. Then the L-graph $\overline{G[C_1 \cup C_2]}$ has a set Z of parallel chords (see Definition 3.7) of size at least 2. Let $v_i v_j$ and $v_{i+1} v_{j'}$ ($\in C_1$) be a pair of parallel chords next to each other with $e^* = v_i v_{i+1} \in C_1 \cap C_2 \subseteq E_{w=2}$ (see Fig. 11(α)). By Claim 5.1, we have that $j' = j - 3$ since, along the segment of $v_{j'} \cdots v_j$ on the circuit C_2, there are two weight 1 edges $v_{j'} v_{j'+1}$, $v_{j-1} v_j$, and one weight 2 edge $v_{j'+1} v_{j-1}$ ($\in C_2 \cap C_3$). Hence,

> $v_i v_{i+1}$ ($\in E_{w=2}$) *is contained in a* 6*-circuit* $v_i v_{i+1} v_{j-3} v_{j-2} v_{j-1} v_j v_i$ *in which weight* 2 *edges are* $v_i v_{i+1}$, $v_{j-2} v_{j-1}$ *and weight* 1 *edges induce two* 2*-paths*: $v_{i+1} v_{j-3} v_{j-2} \subseteq Q_i$, $v_{j-1} v_j v_i \subseteq Q_j$, *for some* $\{i, j\} = \{1, 2\}$.

(In Fig. 11(β), Q_1 and Q_2 are the two chordless circuits of the permutation graph G.)

We will show that the edge $e_0 = x_0 y_0$ is not contained in any circuit of the type described in the previous paragraph. (See Fig. 12.) Assume that $x_0 u_1 u_2 w_2 w_1 y_0 x_0$ is such a 6-circuit of G contain-

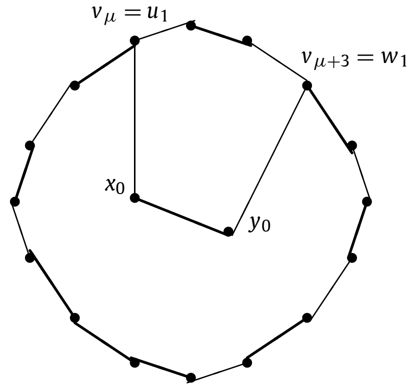


Fig. 12. A 6-circuit containing $e_0 = x_0 y_0$?

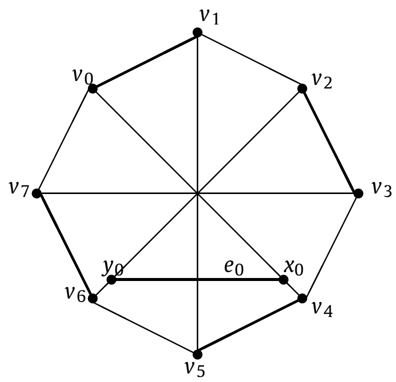


Fig. 13. The Petersen graph ($h = 2$).

ing the edge $x_0 y_0$ with $x_0 u_1 u_2 \subseteq Q_1 \subseteq E_{w=1}$, $y_0 w_1 w_2 \subseteq Q_2 \subseteq E_{w=1}$ and $x_0 y_0, u_2 w_2 \in E_{w=2}$. Since x_0 and y_0 are vertices inserted into the diagonal crossing chords of L-graphs $H_{1,2}$ and $H_{2,3}$, edges $x_0 u_1, y_0 w_1$ are part of the diagonal crossing chords. Let $u_1 = v_\mu$ and $w_1 = v_\nu$. Furthermore, edges $u_1 u_2, w_1 w_2 \in E_{w=1}$ are all in C_2. Thus, $u_2 = v_{\mu\pm1}$, $w_2 = v_{\nu\pm1}$, and the edge $u_2 w_2 \in E_{w=2}$ is also in C_2. That is, $u_1 u_2 w_2 w_1$ is a segment of the circuit C_2, say $u_1 u_2 w_2 w_1 = v_\mu v_{\mu+1} v_{\mu+2} v_{\mu+3}$.

Note that $v_{\mu-1} v_\mu \in C_1 \cap C_2$ since $v_\mu x_0 \in C_1$ while $v_\mu v_{\mu+1} \in C_2 - C_1$. So, by Claim 5.1, $v_{\mu+1} v_{\mu+2} \in C_3 \cap C_2$.

Similarly, $v_{\mu+3} v_{\mu+4} \in C_3 \cap C_2$ since $v_{\mu+3} y_0 \in C_3$ while $v_{\mu+2} v_{\mu+3} \in C_2 - C_3$. However, by Claim 5.1, $v_{\mu+1} v_{\mu+2} \in C_1 \cap C_2$. This is a contradiction.

Again, the edge $e_0 = x_0 y_0$ was selected *arbitrarily*. If we consider the admissible pair $(G - e^*, w)$ instead of $(G - e_0, w)$, the vertices v_i and v_{i+1} are now at the positions as that of x_0 and y_0. However, we have shown that the edge $e_0 = x_0 y_0$ is not contained in any 6-circuit described above. This completes the proof of Claim 5.2 that $h = 2$.

The final step: the Petersen graph. By Claim 5.2, there is only one remaining case: $h = 2$. That is, both $H_{1,2}$ and $H_{2,3}$ are K_4. By Claim 5.1, it is easy to see that G is the Petersen graph. (Note that, in Fig. 13, bold lines are edges in $E_{w=2}$ and slim lines are edges in $E_{w=1}$.)

6. Proofs of some lemmas in Section 3.3

6.1. Basic lemmas

Although the following lemmas are rather trivial and straightforward, they are very useful and will be frequently used in some induction proofs.

Lemma 6.1. *(See [10].) Let $G \in \langle \mathcal{K}_4 \rangle$. If $|V(G)| \geqslant 6$ then G contains at least two triangles, and all triangles are mutually disjoint with each other.*

Lemma 6.2. *Let $G \in \langle \mathcal{K}_4 \rangle$. If G is of order at least 4, then G must be 3-connected.*

Lemma 6.3. *Let S be a triangle of a cubic graph G. Then $G \in \langle \mathcal{K}_4 \rangle$ if and only if the contracted graph $G/S \in \langle \mathcal{K}_4 \rangle$.*

Proof. The "if" part is obvious by Definition 1.4. Here, we only prove the "only if" part.

Induction on $|V(G)| = 2n$. It is trivial if $G = 3K_2$ and $G = K_4$. Let G be obtained from $3K_2$ via the following sequence of $(Y \to \triangle)$-operations: $\pi_1, \ldots, \pi_{n-1}$. Let S^* be the triangle of G that is created by the last $(Y \to \triangle)$-operation π_{n-1}. By Definition 1.4, $G/S^* \in \langle \mathcal{K}_4 \rangle$ since it is obtained from $3K_2$ via the sequence of $(Y \to \triangle)$-operations: $\pi_1, \ldots, \pi_{n-2}$.

If $S^* = S$, then $G/S = G/S^* \in \langle \mathcal{K}_4 \rangle$. Hence, we assume that $S^* \neq S$. Since $G/S^* \in \langle \mathcal{K}_4 \rangle$ and is smaller than G, by inductive hypothesis, we have that $(G/S^*)/S \in \langle \mathcal{K}_4 \rangle$. By Definition 1.4, $(G/S^*)/S$ is obtained from $3K_2$ via a sequence of $(Y \to \triangle)$-operations: $\pi_1^*, \ldots, \pi_{n-3}^*$. Note that $(G/S^*)/S = (G/S)/S^*$ since S and S^* are disjoint (by Lemma 6.1). So, G/S, is obtained from $3K_2$ via the sequence of $(Y \to \triangle)$-operations: $\pi_1^*, \ldots, \pi_{n-3}^*, \pi_{n-1}$. □

Lemma 6.4. *Let $G \in \langle \mathcal{K}_4 \rangle$ with $|V(G)| \geqslant 6$ and let S be a triangle of G and let T be a 3-edge-cut of G. If $E(T) \cap E(S) = \emptyset$, then T remains as a 3-edge-cut of the contracted graph G/S.*

Lemma 6.5. *Let $G \in \langle \mathcal{K}_4 \rangle$ with $|V(G)| \geqslant 6$ and $e \in E(G)$ and let S be a triangle of G. If $e \notin E(S)$ but is incident with some vertex of S, then e is not contained in any triangle of G.*

Proof. By Lemma 6.1. □

Lemma 6.6. *Let $G \in \langle \mathcal{K}_4 \rangle$ and $w : E(G) \mapsto \{1, 2\}$ be a Hamilton weight of G and let $\{C_1, C_2\}$ be the faithful circuit cover of (G, w) with both C_j as a Hamilton circuit. If G has a triangle S, then $\{C_1/S, C_2/S\}$ is a faithful cover of $(G/S, w)$ and $(G/S, w)$ remains a Hamilton weight pair.*

Proof. By Lemma 6.3, we have that $G/S \in \langle \mathcal{K}_4 \rangle$. Let T be the set of edges incident with S but not in S. Since both C_1 and C_2 are Hamilton circuits of G, each of them passes through the 3-edge-cut T precisely *twice*. Let $T = \{f_0, f_1, f_2\}$ and $f_0, f_i \in E(C_i)$, and let $E(S) = \{e_{0,1}, e_{0,2}, e_{1,2}\}$ where $e_{\mu,\nu}$ is adjacent with f_μ and f_ν. It is obvious that $w(f_0) = w(e_{1,2}) = 2$ and all others are of weight 1. Thus, $C_i/S = C_i/\{e_{0,(3-i)}, e_{i,(3-i)}\}$ remains as a Hamilton circuit in the contracted graph G/S, and, therefore, $\{C_1/S, C_2/S\}$ is a faithful cover of G/S consisting of two Hamilton circuits.

In the remaining part of the proof, we are to show that $(G/S, w)$ is a *Hamilton weight pair*. That is, *every faithful circuit cover of $(G/S, w)$ consists of a pair of Hamilton circuits*. Suppose that $\mathcal{C}_S$ is a faithful cover of $(G/S, w)$ consisting of at least three members. Then, let $D_i \in \mathcal{C}_S$ that $f_0, f_i \in E(D_i)$. Extend D_i to a circuit in G by adding edges $e_{0,(3-i)}, e_{i,(3-i)}$ of S, $\mathcal{C}_S$ becomes a faithful cover of (G, w). This contradicts that (G, w) is a Hamilton weight pair and completes the proof of the lemma. □

Lemma 6.7. *Let $G \in \langle \mathcal{K}_4 \rangle$ of order at least 4. If T is a 3-edge-cut of G and contains some edge of a triangle S, then T must be a trivial 3-edge-cut of G that contains two edges of the triangle.*

Proof. It is trivial that $|S \cap T|$ must be even since one is a circuit while another one is a cut. And $|S \cap T| = 2$ since $|E(S)| = 3$. The cut T is trivial since it contains two edges of $E(v)$ for some vertex $v \in V(S)$. □

Lemma 6.8. *Let $G \in \langle \mathcal{K}_4 \rangle$ and $w : E(G) \mapsto \{1, 2\}$ be an eulerian weight of G. Then w is a Hamilton weight of G if and only if $E_{w=1}$ induces a Hamilton circuit of G.*

Proof. Use Lemmas 6.1 and 6.3 for induction (contracting a triangle). □

6.2. Proof of Lemma 3.6

Induction of $|V(G)|$. It is trivial that the lemma is true for K_4. Hence, we assume that $|V(G)| \geqslant 6$. By Lemma 6.1, G contains at least two disjoint triangles.

For statements (1) and (2).

I. We claim that *there is a triangle S in G such that e is not incident with any vertex of S.*

Suppose that the edge e is incident with *every* triangle of G. By Lemmas 6.1 and 6.5, let $e = x_1y_1$, and $S_1 = x_1x_2x_3x_1$, $S_2 = y_1y_2y_3y_1$ which are the only two triangles of G. Note that, the set T of all edges incident with S_1 but not in S_1 is a non-trivial 3-edge-cut of G. This contradicts the assumption that every 3-edge-cut of G containing e is trivial.

II. By I, let S_1 be a triangle of G such that the edge e is not incident with any vertex of S_1. Note that, by Lemma 6.3, $G/S_1 \in \langle \mathcal{K}_4 \rangle$ and satisfies the description of the lemma since the contraction of S_1 does not create any new 3-edge-cut of G/S_1. By induction, statements (1) and (2) hold for G/S_1. That is,

the suppressed cubic graph $\overline{G/S_1 - e} \in \langle \mathcal{K}_4 \rangle$ and $\{e_1, e_2\}$ is contained in a 3-edge-cut T' of G/S_1.

Note that $(\overline{G-e})/S_1 = \overline{G/S_1 - e} \in \langle \mathcal{K}_4 \rangle$ which implies that $\overline{G-e} \in \langle \mathcal{K}_4 \rangle$ since $\overline{G-e}$ is obtained from $(\overline{G-e})/S_1 = \overline{G/S_1 - e} \in \langle \mathcal{K}_4 \rangle$ via a $(Y \to \Delta)$-operation. This proves statement (1).

III. Continue from II, the 3-edge-cut T' of $(\overline{G-e})/S_1$ containing the edges e_1, e_2 remains as a 3-cut in $\overline{G-e}$ (after a $(Y \to \Delta)$-operation) since e is not incident with any vertex of S_1. statement (2) is therefore verified.

For statement (3). Statement (3) is an immediate corollary of statement (1) and Lemma 6.8.

6.3. Proof of Lemma 3.8

Induction on n where $|V(G)| = 2n$. It is obviously true if $n = 2$. By Lemma 6.1, the graph G with at least 6 vertices has at least two disjoint triangles.

I. We claim that

(1) *e is not contained in any triangle of G,* and
(2) *e is incident with every triangle of G.*

Since G has at least two disjoint triangles, let $S = z_1z_2z_3z_1$ be a triangle of G such that $e \notin S$. Let x_i be the neighbor of z_i not contained in the triangle S. By hypothesis (a), assume that $z_1z_2 \in F$ and $z_3z_1, z_2x_2 \in E_{w=2}$. (See Fig. 14.)

By Lemma 6.7, every 3-edge-cut of G containing z_1z_2 must be *trivial* since z_1z_2 is contained in the triangle S. Hence, e must be one of those two edges $\{z_2z_3, z_1x_1\}$ (since $\{z_2z_3, z_1x_1\} = (E(z_1) \cup E(z_2)) \cap E_{w=1} - \{z_1z_2\}$). Since $e \notin S$, we have that $e = z_1x_1$ in G. By Lemma 6.5, the edge e is not contained in any triangle of G. With the same argument as above, e is incident with every triangle of G.

II. By I(2), the graph G has precisely two triangles since e has precisely two endvertices and every triangle of G must contain one of these two.

III. Consider the smaller graph $G' = G/S$ with s as the new vertex created by the contraction. Note that e is incident with s. Add sx_3 into F (if it was not in F). (See Fig. 14.)

IV. We claim that, *for the smaller graph G', the new admissible pair (G', w) has all the properties described in the lemma.*

By Lemmas 6.3 and 6.6, the graph $G' \in \langle \mathcal{K}_4 \rangle$ and (G', w) remains as a Hamilton weight pair, and the pair of Hamilton circuits $\{C_1/S, C_2/S\}$ of G' is a faithful cover of (G', w).

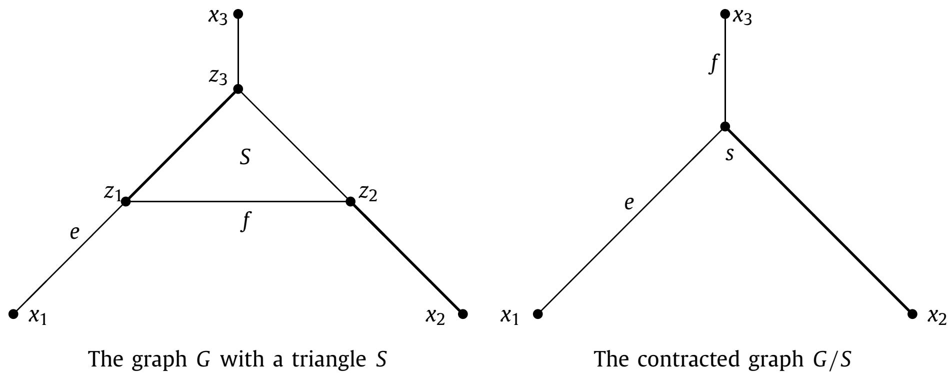


Fig. 14. Induction proof of Lemma 3.8.

For the description (a), any new triangle created by contraction must contain the new vertex s, and therefore, must contain either e or the newly-labeled F-edge x_3s. For the description (b), for any F-edge f other than x_3s, (by Lemma 6.4) a 3-cut of G containing f and e remains as 3-cut in the smaller graph G'; for the edge x_3s, the trivial cut $E(s)$ of $G/S = G'$ contains both x_3s and e.

Hence, by induction, (G', w) is an L-graph pair with e as the diagonal crossing chord. Obviously, (G, w) is also an L-graph pair since the diagonal chord e is incident with s at which G is obtained from G' via a $(Y \to \Delta)$-operation.

6.4. *Untouched edges of $\langle \mathcal{K}_4 \rangle$-graphs*

Definition 6.9. Let $G \in \langle \mathcal{K}_4 \rangle$ of order at least 4. Classify edges of G into two groups: edges, e, such that $\overline{G - e}$ remains 3-connected are labeled as *"untouched"*, and all others are labeled as *"touched"*.

It is trivial that edges satisfying Lemma 3.6 are labeled as *"untouched"* while all others are labeled as *"touched"*.

Since G is obtained from K_4 via a series of $(Y \to \Delta)$-operations (by Definition 1.4), the classification of edges of G (Definition 6.9) can be viewed as follows.

Lemma 6.10. *For a $\langle \mathcal{K}_4 \rangle$-graph G of order at least 4, and for a series Π of $(Y \to \Delta)$-operations that constructs the graph G from K_4, an edge $e \in E(G)$ is labeled as "touched" if one endvertex of e is expanded to a triangle by at least one $(Y \to \Delta)$-operation of Π. All edges that are not labeled as "touched" in any $(Y \to \Delta)$-operation during the construction are labeled as "untouched".*

Observation. Let $G \in \langle \mathcal{K}_4 \rangle$ of order at least 4. Every edge contained in some triangle is untouched, while an edge incident with some triangle but not in any triangle is "touched". We also notice that not every untouched edge is contained in some triangle.

Lemma 3.6 can be further generalized as follows.

Lemma 6.11. *Let $G \in \langle \mathcal{K}_4 \rangle$ of order at least 4, and $e = v_1v_2 \in E(G)$. For each $i = 1, 2$, let e_i be the edge of $\overline{G - e}$ containing the vertex v_i (that is, e_i is an attachment of e in the suppressed graph $\overline{G - e}$). Then the following statements are equivalent.*

(1) *Every 3-edge-cut of G containing e is trivial (that is, $E(v_1)$ and $E(v_2)$ are the only two 3-edge-cuts of G containing the edge e);*
(2) *The suppressed cubic graph $\overline{G - e} \in \langle \mathcal{K}_4 \rangle$;*
(3) *$\{e_1, e_2\}$ is contained in a 3-edge-cut of $\overline{G - e}$.*

Lemma 6.12. *Let $G \in \langle \mathcal{K}_4 \rangle$, let w be a Hamilton weight of G, and $e \in E_{w=2}$. Then*

(1) *$(G - e, w)$ has a unique faithful circuit cover $\mathcal{C}$;*
(2) *$|\mathcal{C}| = 1$ if and only if $G = 3K_2$;*

(3) $|\mathcal{C}| = 2$ *if and only if e is untouched*;
(4) $|\mathcal{C}| \geqslant 3$ *if and only if e is touched.*

The proofs of Lemmas 6.10, 6.11 and 6.12 are omitted in this paper since they are not used in the proof of Theorem 1.9.

7. Remarks

7.1. A note about the conjecture of Fleischner and Jackson

The following is the original version Conjecture 1.5.

Conjecture 7.1. *(See Fleischner and Jackson, Conjecture 12 in [5] or see [8].) Let (G, w) be a contra pair such that the set of all weight 1 edges induces a 2-factor consisting of a pair of chordless circuits Q_1 and Q_2, and the set of all weight 2 edges induces a perfect matching M joining Q_1 and Q_2. If (G, w) has no removable circuit, then $(G, w) = (P_{10}, w_{10})$ (see Fig. 4).*

Here, we are to show that Conjecture 1.5 and Conjecture 7.1 are equivalent to each other.

By Lemma 4.1, a minimal contra pair has the structure described in Conjecture 7.1. So it is sufficient to show that the contra pair described in Conjecture 7.1 is a minimal contra pair. That is, for every edge $e \in E_{w=2}$, we are to show that $(G - e, w)$ has a faithful cover.

Let (G, w) be the admissible pair described in Conjecture 7.1. If M is of even size then G is 3-edge-colorable. Therefore (G, w) is not a contra pair, and every member of a faithful cover is removable. Hence, M must be of odd size. For every edge $e_0 = xy \in E_{w=2}$, the suppressed cubic graph $\overline{G - e_0}$ is 3-edge-colorable since both Q_1 and Q_2 are of even length in $\overline{G - e_0}$. The 3-edge-coloring of $\overline{G - e_0}$ induces a faithful circuit cover of the admissible pair $(G - e_0, w)$.

With the same argument as above, one is able to see that Conjecture 1.6 is also solved for the family of permutation graphs under the assumption of Hamilton weight conjecture.

7.2. About the main theorem (Theorem 1.9)

The following is a list of some weak versions of Conjecture 1.3 including the statement of Lemma 3.5.

Conjecture 7.2. *Let (G, w) be a Hamilton weight pair. If $E_{w=1}$ induces a Hamilton circuit of G, then $G \in \langle \mathcal{K}_4 \rangle$.*

Conjecture 7.3. *Let (G, w) be an admissible pair, and $e_0 \in E_{w=2}$. Suppose that*

(1) *$E_{w=1}$ induces a Hamilton circuit of G,*
(2) *every removable circuit of (G, w) must contain the edge e_0.*

Then the graph $G \in \langle \mathcal{K}_4 \rangle$.

The relations between those conjectures are presented as follows.

$$\text{Conjecture 1.3} \quad \Rightarrow \quad \text{Conjecture 7.2} \quad \Rightarrow \quad \text{Conjecture 7.3} \quad \Rightarrow \quad \text{Conjecture 1.5}$$

The relation that "Conjecture 7.2 $\Rightarrow$ Conjecture 7.3" can be proved by applying Lemma 3.5; "Conjecture 7.3 $\Rightarrow$ Conjecture 1.5" by using the proof technique of Theorem 1.9.

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