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Triangle-Free Circuit Decompositions and Petersen Minor

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In this paper, we characterize those simple graphs with no Petersen minor which admit triangle-free circuit decompositions. © 1998 Academic Press

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1. INTRODUCTION

A *circuit decomposition* of a graph G is a collection $\mathscr{C}$ of circuits in G such that each edge of G belongs to exactly one of the circuits of $\mathscr{C}$. It is well known that a connected graph has a circuit decomposition if and only if it is eulerian (every vertex has even degree).

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Various circuit decompositions with restrictions on the circuit lengths have been studied. Eulerian-graphs with circuit decompositions without digons were characterized by Seymour [11] for planar graphs and by Alspach, Goddyn, and Zhang [2] for graphs without a Petersen minor. Eulerian graphs with even circuit decompositions were characterized by Seymour [12] for planar graphs and by Zhang [14] for graphs without a K_5-minor.

In [4], Bondy asked "*Which simple graphs admit decompositions into circuits of length at least k*?" When $k=3$, the answer is all eulerian graphs. Assume then that $k \geqslant 4$. In [6] Heinrich, Liu, and Yu characterized graphs of maximum degree 4 which admit triangle-free eulerian tours (that is, no three consecutive edges on the tour form a triangle); but these tours may not result in a triangle-free circuit decomposition. A similar problem was studied by Bertram and Horák [3], and Kouider and Sabidussi [10], who gave sufficient conditions for decomposing 4-regular graphs into triangle-free 2-factors. In this paper, the main result (Theorem 4.1) characterizes all simple graphs with no Petersen minor which admit triangle-free circuit decompositions.

The study of triangle-free decomposition is also motivated by the circuit double cover conjecture (Szekeres [13] and Seymour [11]) that "*Every bridgeless graph has a circuit double cover.*" Let G be a bridgeless graph. We construct an eulerian graph G' from G as follows: for each edge $e=xy$, add a new vertex v_e and a new 2-path xv_ey. It is evident that G has a circuit double cover if and only if the eulerian graph G' has a triangle-free circuit decomposition.

The main result (Theorem 4.1) deals only with graphs with no Petersen minor. For graphs containing a Petersen minor, the result may not be true. Let $M=\{x_iy_i\colon i=1, ..., 5\}$ be a perfect matching of the Petersen graph P_{10}. For each $x_iy_i \in M$, add a 2-path $x_iv_iy_i$ joining x_i and y_i. This new graph does not have a triangle-free decomposition.

2. A FAMILY OF EXCEPTIONAL GRAPHS

We begin with a study of Δ-graphs. It will be shown that these graphs do not have a triangle-free decomposition. Furthermore, as we will see in Section 4, they are precisely the graphs not containing a P_{10}-minor which have no triangle-free decomposition.

Define a family of graphs $\Delta(2i+1)$ as follows: The family $\Delta(1)$ contains only the triangle $x_1x_2x_3$. Suppose we have constructed all graphs in $\Delta(2i-1)$. For each graph H in $\Delta(2i-1)$, choose an edge $e=xy$ (called a **base edge**) of H and, using two new vertices x_{2i} and x_{2i+1}, attach the 4-circuit $x_{2i}xx_{2i+1}y$. In successively choosing base edges no three may ever form a triangle and a base edge may be chosen more than once. All graphs so obtained determine $\Delta(2i+1)$ and are called **$\Delta(2i+1)$-graphs**.

By a **Δ-graph** we mean a $\Delta(2i+1)$-graph for some i. It is obvious that each edge of a Δ-graph is contained in a triangle.

The following results describe the structure of Δ-graphs.

LEMMA 2.1. *Every edge of a $\Delta(2s+1)$-graph lies in a triangle, and if $s \geqslant 1$, in every triangle at least one edge is a base edge.*

LEMMA 2.2. *The set of base edges in a Δ-graph is an invariant of the graph.*

Proof. The claim is clearly true for all graphs in $\Delta(1)$ and $\Delta(3)$. Suppose it is true for all graphs in $\Delta(2(s-1)+1)$ and let $G \in \Delta(2s+1)$. Then G is constructed from a triangle by the addition of 4-circuit and so a set of base edges is defined. Let C be the last 4-circuit added in the construction of G. The graph $G-C$ is a $\Delta(-2(s-1)+1)$ graph and so has a unique set of base edges (those we have already defined). There is a unique edge we can use as a base edge to add C and so obtain G. Thus the base edges used to construct G are unique. ∎

LEMMA 2.3. *A Δ-graph is 2-connected.*

Proof. Clearly $\Delta(1)$-graphs are 2-connected. Suppose all $\Delta(2(s-1)+1)$-graphs are 2-connected. Let G be a $\Delta(2s+1)$-graph which by definition is constructed from a $\Delta(2(s-1)+1)$-graph H by the addition of a 4-circuit C. Suppose G has a cut-vertex x. If $x \in V(H) - V(C)$, then x is a cut-vertex of H, a contradiction. Clearly $x \notin V(C) - V(H)$. So $x \in V(C) \cap V(H)$ and since H and C are both 2-connected, x cannot disconnect G. ∎

LEMMA 2.4. *Let G be a $\Delta(2s+1)$-graph and C_0 be a triangle of G. Then G can be recursively constructed via a series of Δ-graphs: $X_0, X_1, ..., X_s = G$ such that each X_i is a $\Delta(2i+1)$-graph, $X_0 = C_0$ and the union of circuits X_0 and $X_{h+1} \backslash E(X_h)$ (a 4-circuit), for $h = 0, ..., s-1$, is a circuit decomposition of G.*

Proof. The proof is by induction on s. The claim is obvious for $s = 0, 1$. By the definition of Δ-graphs, G is constructed via a series of Δ-graphs $Y_0, Y_1, ..., Y_s = G$, where Y_0 is a triangle and $Y_h \backslash E(Y_{h-1})$ is a 4-circuit for $h = 1, 2, ..., s$.

Let h be the smallest integer such that $C_0 \subset Y_h$. If $h < s$, then by the induction hypothesis, Y_h is constructed via a series of Δ-graphs $C_0 = X_0, ..., X_h$. Let $X_i = Y_i$ for $i = h+1, ..., s$. Then, using Lemma 2.2, the graph G is constructed via the series of Δ-graphs $X_0, ..., X_h, X_{h+1}, ..., X_s$ so that X_i is obtained from X_{i-1} by attaching a 4-circuit.

So $h=s$, and, therefore, the 4-circuit $Y_s \backslash E(Y_{s-1})$ must contain two edges of C_0. Let $Y_s \backslash E(Y_{s-1}) = xx_{2s} y_{x2s+1} x$ with $x, y \in V(Y_{s-1})$. Without loss of generality, let $C_0 = xx_{2s}yx$. By the induction hypothesis, Y_{s-1} can be constructed via a series of Δ-graphs $Z_0, Z_1, ..., Z_{s-1}$ so that the edge xy is in the triangle $Z_0 = xyzx$ (Lemma 2.1). Let $X_0 = xyx_{2s}x = C_0$, $X_1 = X_0 \cup \{xzyx_{2s+1}x\}$, and $X_h = X_1 \cup Z_{h-1}$ for $h = 2, ..., s$. Then, again by Lemma 2.2, G is constructed via the series of Δ-graphs $X_0, ..., X_s$. ▮

LEMMA 2.5. *No Δ-graph has a triangle-free circuit decomposition.*

Proof. If a $\Delta(2s+1)$-graph G has only one base edge e, then G is $K_{1,1,2s+1}$ and as any circuit containing e is a triangle, there is no triangle-free circuit decomposition.

Now suppose that G has at least two base edges and consider the first two base edges $e_1 = v_1 u$ and $e_2 = uv_2$, used in some recursive construction of G. Then there is an edge $e_3 = v_1 v_2$ so that $v_1 v_2 u v_1$ is a triangle of G. Certainly, e_3 is not a base edge and therefore u is a cut-vertex of $G \backslash \{e_3\}$. By Lemma 2.4, G is constructed via a series of Δ-graphs $\{X_0, X_1, ..., X_s\}$ where $X_0 = uv_1 v_2 u$ and $Z_i = X_i \backslash X_{i+1}$ is a circuit of length 4.

Let H_1 and H_2 be subgraphs of G defined recursively as follows. Starting at $i=0$, let $H_1 = H_2 = X_0$. For each $i = 1, 2, ..., s$, if Z_i contains two vertices of $H_\mu \backslash \{v_\nu\}$ for $\mu, \nu \in \{1, 2\}$ and $\mu \neq \nu$, replace H_μ by $H_\mu \cup Z_i$.

Then both H_1 and H_2 are Δ-graphs, and $H_1 \cap H_2$ is the triangle X_0. Note that both H_1 and H_2 are strictly smaller than G.

Suppose that G is a smallest Δ-graph which has a triangle-free circuit decomposition $\mathscr{C}$. Then there is a circuit C in $\mathscr{C}$ containing e_3 and u but not both e_1 and e_2. Suppose e_2 is not in C. Then replacing the path segment of C in H_1 by the path uv_1v_2 gives a circuit C' of H_2 with at least four edges. C', together with the set of circuits of $\mathscr{C}$ which are entirely contained in H_2, form a triangle-free circuit decomposition of H_2. This contradicts the fact that G is a smallest counterexample. Therefore, G does not have a triangle-free circuit decomposition. ▮

The circuit decomposition $\mathscr{F}$ of G described in Lemma 2.4 will be called **Δ-decomposition with triangle** X_0.

We now study a graph very similar to a Δ-graph, except that it has a triangle-free circuit decomposition. This graph will be used in the proof of the main theorem.

A graph is called a pseudo-Δ-graph if it is constructed in the same way as a Δ-graph, except that base edges may form a triangle and there is at least one triangle in which each edge is a base edge.

LEMMA 2.6. *Let G be a pseudo-$\Delta(2s+1)$-graph and C_0 be a triangle of G. Then G can be recursively constructed via a series of Δ-graphs or* *pseudo-Δ-graphs*: $X_0, X_1, ..., X_s = G$ *such that each* X_i *is a (pseudo-)* $\Delta(2i+1)$-*graph, and the union* $X_0 = C_0$ *and* $X_{h+1} \backslash E(X_h)$ *(a 4-circuit) for* $h = 0, ..., s-1$ *is a circuit decomposition with* X_0 *the only triangle.*

Proof. The proof is analogous to that of Lemma 2.4. ∎

LEMMA 2.7. *Every pseudo Δ-graph has a triangle-free circuit decomposition.*

Proof. The smallest pseudo-Δ-graph is the pseudo-$\Delta(7)$-graph and it has a triangle-free circuit decomposition.

Now let G be a pseudo-$\Delta(2s+1)$-graph, where $2s+1 > 7$. Let X_0 be a triangle of G which contains three base edges. Then by Lemma 2.6, we can assume that G is recursively constructed via a series of pseudo-Δ-graphs $X_0, X_1, ..., X_s = G$. Furthermore, we can choose the sequence so that X_3 is the pseudo-$\Delta(7)$-graph. Let $\mathscr{C}_1$ be a triangle-free decomposition of X_3. Then $\mathscr{C} = \mathscr{C}_1 \cup \{X_h \backslash E(X_{h-1}) \mid h = 4, ..., s\}$ is a triangle-free circuit decomposition of G. ∎

3. REMOVABLE CIRCUITS

DEFINITION 3.1. Let G be a graph. A circuit C in G is removable if $G \backslash E(C)$ is the union of a 2-connected graph and possibly some isolated vertices.

LEMMA 3.2 (Goddyn, van den Heuvel, and McGuinness [5]) (or see [15, p. 270] *where the removable circuit theorem is presented as one section of the book*). *Let G be a* 3-*connected eulerian graph. If G has a circuit decomposition* $\mathscr{F}$ *such that each member of* $\mathscr{F}$ *is a circuit of length at least* 3, *then* $\mathscr{F}$ *contains two edge-disjoint removable circuits of G.*

LEMMA 3.3 (Goddyn, van den Heuvel, and McGuinness [5]). *Let G be a* 3-*connected eulerian graph. If G contains no subdivision of the Petersen graph, then G has a circuit decomposition* $\mathscr{F}$ *which contains two edge-disjoint removable circuits of G, each having length at least three.*

4. THE MAIN THEOREM

Our main result is the following.

THEOREM 4.1. *Let G be a simple* 2-*connected eulerian graph with no* P_{10}-*minor. Then G admits a triangle-free circuit decomposition if and only if G is not a* Δ-*graph.*

The proof of necessity in this theorem follows from Lemma 2.5. The remainder of the paper proves the sufficiency. The proof is by contradiction and begins with an investigation of thesproperties of the smallest possible counterexample.

LEMMA 4.2. *Let G be a smallest counterexample to the sufficiency statement of Theorem* 4.1. *Then, for every 2-vertex-cut $J=\{x, y\}$ (if it exists), exactly one component of $G \backslash J$ is a single vertex and $xy \in E(G)$.*

Proof. Let G be a smallest simple 2-connected eulerian graph with no Peterson minor which is not a Δ-graph and has no triangle-free decomposition. Then any 2-connected eulerian subgraph of G which is not a Δ-graph will have a triangle-free decomposition. Let $J=\{x, y\}$ be a 2-vertex-cut of G that "separates" G into two connected graphs G_1 and G_2, where $G=G_1 \cup G_2$ and $V(G_1) \cap V(G_2)=J$.

Suppose first that xy is not an edge of G. It is easy to see that $d(x)$ and $d(y)$ have the same parity in G_i $(i=1, 2)$. Let H_i be the eulerian graph obtained from G_i by adding either the edge xy or a triangle $T_i=xyv_ix$ (where v_i is a new vertex) depending on the parity of the degree of x in G_i $(i=1, 2)$. Then both H_1 and H_2 are smaller than G and are not counterexamples to the theorem. Let $\mathscr{F}_i$ be a circuit decomposition of $H_i (i=1, 2)$ such that for each $i \in \{1, 2\}$:

(1) if H_i is not a Δ-graph, then $\mathscr{F}_i$ is a triangle-free circuit decomposition of H_i;

(2) if H_i is a Δ-graph, then $\mathscr{F}_i$ is a Δ-decomposition with triangle C_i^*, where C_i^* contains the new edge xy or is the new triangle T_i. In the latter case, we let $C_i^{**} \in \mathscr{F}_i$ be a 4-circuit incident with both x and y.

We consider the various possibilities for H_1 and H_2.

If both G_1 and G_2 are eulerian, there are three possibilities for H_1 and H_2. If H_1 and H_2 are Δ-graphs, then $C_i^*=T_i$, and $[\mathscr{F}_1 \backslash \{C_1^*\}] \cup [\mathscr{F}_2 \backslash \{C_2^*\}]$ is a triangle-free circuit decomposition of G. If only H_1 is a Δ-graph, then $\mathscr{F}_1 \backslash \{C_1^*\}$ is a triangle-free circuit decomposition of G_1 and $\mathscr{F}_2 \backslash \{T_2\}$ is a decomposition of H_2 into circuits of length at least 4 and two paths of length at least 2 with endpoints x and y. Adding the edges of C_1^* to these paths we obtain a triangle-free circuit decomposition of G. Finally, if neither H_1 nor H_2 is a Δ-graph, joining the two paths from each of $\mathscr{F}_1 \backslash \{T_1\}$ and $\mathscr{F}_2 \backslash \{T_2\}$ results in a triangle-free circuit decomposition of G.

If both G_1 and G_2 are not eulerian, the degrees of x and y are odd in G_1 and G_2, and H_i is obtained from G_i by adding the edge xy. Then $\mathscr{F}_i \backslash \{xy\}$ is a decomposition of H_i into circuits of length at least 4 and a path of length at least 2 with endpoints x and y. Joining the paths together gives a triangle-free decomposition of G.

Therefore it must be the case that xy is an edge of G. We will assume that x and y have odd degrees in both G_1 and G_2, by $xy \in E(G_1)$ and $xy \notin E(G_2)$. Let H_1 be the graph obtained from G_1 by adding a new vertex v and two new edges xv and yv; and H_2 the graph obtained from G_2 by adding the edge xy. We choose $\mathscr{F}_i$ to be a circuit decomposition of H_i as described in (1) and (2), provided $H_1 \ncong G$ (clearly $H_2 \ncong G$).

If at least one of the $\mathscr{F}_i$ is triangle-free, delete the edges of $H_i \backslash G_i$ and join the resulting paths to obtain a triangle-free circuit decomposition of G. Therefore we can assume both $\mathscr{F}_i$ contain triangles. Since both H_1 and H_2 are smaller than G, they must be Δ-graphs. Let H_1 be constructed via the series of Δ-graphs $X_0, ..., X_r$, with $X_0 = xyvx$, and H_2 be constructed via the series $Y_0, ..., Y_s$, with $Y_0 = xyzx$. Then it is easily seen that G is a Δ-graph or a pseudo-Δ-graph constructed via the series of δ- and pseudo-Δ-graphs $Z_0, ..., Z_{r+s}$, where $Z_i = Y_i$ for $i = 0, 1, ..., s$ and $Z_{i+s} = [X_i \backslash E(X_0)] \cup H_2$ for $i = 1, ..., r$ and, hence, G has a triangle-free circuit decomposition or G is a Δ-graph; a contradiction in either case.

Thus, $H_1 \cong G$ and, since G is not a Δ-graph, for every 2-vertex-cut J exactly one component of $G - J$ is an isolated vertex. Note that, should $G - J$ contain two or more isolated vertices, simply choose G_1 so that at least two of them lie in it and obtain a decomposition or discover that G must be a Δ-graph. ▮

A triangle $xyzx$ in G its called a **pendant triangle** if $d(x), d(y) \geqslant 4$ and $d(z) = 2$ (implying $\{x, y\}$ is a 2-vertex-cut of G). The vertices x and y are called the **attachments** of the pendant triangle.

Lemma 4.2 allows us to immediately conclude:

LEMMA 4.3. *Every pair of distinct pendant triangles in a smallest counter-example to the sufficiency of Theorem* 4.1 *are edge-disjoint.*

We will use the following notation in the proof of our main result:

For a Δ-graph G, let G be constructed via a series of Δ-graphs $X_0, ..., X_q$ and let $\mathscr{F} = \{C_0, ..., C_q\}$ be the corresponding Δ-decomposition with triangle $X_0 = C_0$ of G. Define the directed graph $\Delta(\mathscr{F})$ as follows: $V(\Delta(\mathscr{F})) = \mathscr{F}$ and (C_i, C_j) is a directed arc in $\Delta(\mathscr{F})$ if and only if $C_j = X_j \backslash E(X_{j-1})$ is a 4-circuit with base edge contained in $C_i (i < j)$. Clearly $\Delta(\mathscr{F})$ is rooted tree with the root C_0. Each vertex of $\Delta(\mathscr{F})$ with zero outdegree is called a **leaf**.

Proof of the sufficiency of Theorem 4.1. Suppose there is a graph satisfying the sufficiency conditions of Theorem 4.1 but which does not have a triangle-free circuit decomposition. Then there is a smallest such graph G. That is, G is a smallest simple 2-connected eulerian graph with no Peterson minor which is not a Δ-graph and has no triangle-free decomposition. Moreover, any 2-connected eulerian subgraph of G which is not a Δ-graph will have a triangle-free decomposition.

For each removable circuit D of G, denote the 2-connected component of $G \backslash E(D)$ by H_D and denote by $\mathscr{F}_D$ a triangle-free circuit decomposition of H_D if H_D is not a Δ-graph, and a Δ-decomposition if H_D is a Δ-graph.

Let G' be the graph obtained from G by contracting one edge of each subdivided edge (if any). By Lemma 4.2, G' has no 2-vertex-cut.

Therefore G' is 3-connected and the multiplicity of each edge is at most 2. By Lemma 3.3, G' has two edge-disjoint removable circuits of length at least three. The corresponding-circuits in G are also removable.

We claim that G has a removable circuit of length at least four and will now verify this. Consider a removable circuit D'_1 of length three in G'. Let D_1 be the corresponding circuit in G and assume D_1 has length 3 (or we are done). We consider two cases depending on whether or not H_{D_1} is a Δ-graph.

Suppose first that H_{D_1} is not a Δ-graph. Since G is a smallest counterexample, H_{D_1} has a triangle-free circuit decomposition $\mathscr{F}_{D_1}$. We now apply Lemma 3.2 to find another removable circuit of G of length at least four contained in $\mathscr{F}_{D_1}$, noting that $\mathscr{F}_{D_1} \cup \{D_1\}$ is a circuit decomposition of G. Let $\mathscr{F}'_{D_1} \cup \{D'_1\}$ be the corresponding circuit decomposition of G', which is digon-free and hence has two removable circuits by Lemma 3.2. Therefore, $\mathscr{F}_{D_1} \cup \{D_1\}$ has two removable circuits of G, one of which is in $\mathscr{F}_{D_1}$ and is of length at least four since H_{D_1}, is simple and $\mathscr{F}_{D_1}$, is triangle-free.

Second, suppose H_{D_1} is a Δ-graph. Let $\mathscr{F}_{D_1} = \{C_0, ..., C_p\}$ be a Δ-decomposition of H_{D_1} with the triangle C_0 intersecting D_1 in a vertex v. Each of the 4-circuits corresponding to a leaf of $A(\mathscr{F}_{D_1})$ must intersect D_1, for otherwise we would contradict Lemma 4.3 and so $A(\mathscr{F}_{D_1})$ has at most two leaves. If $A(\mathscr{F}_{D_1})$ has two leaves, each is a removable circuit of G since Δ-graphs are 2-connected (Lemma 2.3). If $A(\mathscr{F}_{D_1})$ has only one leaf, then $A(\mathscr{F}_{D_1}) = C_0 \cdots C_p$ is a directed path, provided $p \geqslant 2$, C_1 is removable in G. If $p = 1$, then $G = K_5$ and is not a counterexample to the theorem.

Next, observe that for each removable circuit D of G of length at least four, H_D must be a Δ-graph as if not, H_D has a triangle-free decomposition $\mathscr{F}_D$, and $\mathscr{F}_D \cup \{D\}$ is a triangle-free decomposition of G, a contradiction. So H_D is a Δ-graph and

$$|E(H_D)| = |E(G) \backslash E(D)|$$

is odd. Thus, the lengths of any two removable circuits with lengths at least four must have the same parity.

We now show that there exist removable circuits of lengths 4 and 5; so obtaining a contradiction and allowing us to conclude that there can be no counterexample to the necessity of Theorem 4.1.

Given D_2, a removable circuit of length at least four, and the fact that H_{D_2} is a Δ-graph, we repeat the above argument to find a second (disjoint) removable circuit of length at least four in G. In fact, that argument shows us that there is such a circuit, D_3, of length precisely 4.

Let $\mathcal{F}_{D_3} = \{C'_0, ..., C'_p\}$ be a Δ-decomposition of H_{D_3} in which the triangle C'_0 intersects D_3.

Suppose the length of a longest directed path in $A(\mathcal{F}_{D_3})$ is at least two. Let $C'_q = v_1 v_2 v_3 v_4 v_1$ be a leaf of $A(\mathcal{F}_{D_3})$ and $C'_{q-1} = v_1 v_3 v_5 v_6 v_1$ be the leaf of $A(\mathcal{F}_{D_3})$ dominating C'_q. Since C'_q must have a vertex in common with D_3 (to avoid contradicting Lemma 4.3), let $v_2 \in V(D_3)$. Then $D_4 = v_1 v_4 v_3 v_5 v_6 v_1$ is a removable circuit of length five in G.

Therefore, the length of a longest directed path in $A(\mathcal{F}_{D_3})$ is one, and all base edges of H_{D_3} are contained in C_0. Each leaf of $A(\mathcal{F}_{D_3})$ must contain a vertex of D_3; as must C_0. Since $|V(D_3)| = 4$ and Lemmas 4.2 and 4.3 hold for the simple graph G, it follows that G is $K_{1,1,5} \cup W$, where the 4-circuit W is on the vertices in the part of $K_{1,1,5}$ of size 5, or G has eight vertices 1, 2, 3, 4, 5, 6, 7, 8 and edges 18, 12, 13, 14, 15, 16, 83, 84, 85, 86, 87, 23, 34, 45, 56. In the first case G has a removable circuit of length 5 and in the second case it has a removable circuit 12341 of length 4.

This completes the proof. ▮

5. REMARKS

We saw in the first section an example of a graph containing a subdivision of the Petersen graph but having no triangle-free circuit decomposition. In fact, for any integer $t \geqslant 2$, we can construct eulerian graphs having no t-gon free circuit decomposition.

Let G be a bridgeless graph and $w: E(G) \mapsto \{1, 2\}$ be a weighting of the edges of G. The weighting is **eulerian** if the total weight of each edge cut of G is even. A family $\mathcal{F}$ of circuits of G is called a **faithful circuit cover of** G **with respect to** w if each edge $e \in G$ is contained in precisely $w(e)$ members of $\mathcal{F}$.

Let G be a bridgeless graph and $w: E(G) \mapsto \{1, 2\}$ be an eulerian weighting. Let M be the set of edges with weight 2. Let G' be the unweighted graph obtained from G by adding a path of length $t-1$ joining each pair of end vertices of $e \in M$. Obviously, the property "G' has a logon free circuit decomposition" implies that "G has a faithful circuit cover with respect to w."

It is known that there exist bridgeless graphs which do not have a faithful circuit cover with respect to some eulerian weighting. Let M be a perfect matching of the Petersen graph P_{10} and w_{10} a weighting of the edges of P_{10} such that $w(e)=1$ for $e \notin M$ and $w(e)=2$ for $e \in M$. It was observed in [8] (also see [11, 1, 2]) that P_{10} does not have a faithful circuit cover with respect to w_{10}. In [9], starting from the above (1,2)-weighted Petersen graphs, by applying the dot product method [7], Jackson constructed a family of infinitely many 3-edge-connected, cyclically 4-edge-connected, cubic graphs each of which does not have a faithful circuit cover. In addition, the edges with weight two form a perfect matching of the graphs. If we attach a triangle to each edge in the perfect matching, we obtain infinitely many eulerian graphs with degrees 2 and 4, which are not Δ-graphs and do not have triangle-free circuit decomposition, and all have the Petersen graph as a minor.

On studying the structure of the counterexamples mentioned, we are led to ask the following question.

PROBLEM 5.1. Characterize the (2, 4)-graphs (each vertex is of degree either 2 or 4) which have triangle-free circuit decompositions.

We note that in [10] it was proven that if all maximal induced paths in a (2, 4)-graph G are of odd length, then G has a triangle-free circuit decomposition. So the problem is open only for graphs with such a maximal path of even length.

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