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Multiple weak 2-linkage and its applications on integer flows of signed graphs

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ABSTRACT

For two pairs of vertices x_1, y_1 and x_2, y_2, Seymour and Thomassen independently presented a characterization of graphs containing no edge-disjoint (x_1, y_1)-path and (x_2, y_2)-path. In this paper we first generalize their result to $k \geq 2$ pairs of vertices. Namely, for $2k$ vertices $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$, we characterize the graphs without edge-disjoint (x_i, y_i)-path and (x_j, y_j)-path for any $1 \leq i < j \leq k$. Then applying this generalization, we present a characterization of signed graphs in which there are no edge-disjoint unbalanced circuits. Finally with this characterization we further show that every flow-admissible signed graph without edge-disjoint unbalanced circuits admits a nowhere-zero 6-flow and thus verify the well-known Bouchet's 6-flow conjecture for this family of signed graphs.

1. Introduction

Let $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be an ordered set of vertices of a graph G. The linkage problem is to find vertex-disjoint paths joining some pairs x_i and y_i, while the weak linkage problem is to find edge-disjoint paths joining some pairs x_i and y_i. The following is a summary of some weak linkage problems.

Problem 1.1 (*Weak 2-Linkage Problem*)**.** For $k = 2$, does the graph contain a pair of edge-disjoint paths P_1 and P_2 such that P_μ joins x_μ and y_μ for each $\mu = 1, 2$?

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Problem 1.2 (*Multiple Weak 2-Linkage Problem*)**.** For any integer $k \geq 2$, is there a pair of integers $1 \leq i < j \leq k$ such that the graph G contains a pair of edge-disjoint paths P_i and P_j such that P_μ joins x_μ and y_μ for each $\mu = i, j$?

Problem 1.3 (*Weak Linkage Problem*)**.** For any integer $k \geq 2$, does the graph G contain a set of pairwise edge-disjoint paths $\{P_1, \dots, P_k\}$ such that P_μ joins x_μ and y_μ, for each $\mu \in \{1, \dots, k\}$.

It is evident that both Problems 1.2 and 1.3 are generalizations of Problem 1.1 by simply letting $k = 2$. Problem 1.1 was completely solved by Seymour and Thomassen independently in Theorem 1.4.

Let H be a contraction of G and let $x \in V(G)$. We use $\hat{x}$ to denote the vertex in H which x is contracted into.

Theorem 1.4 (*Seymour [15], Thomassen [18]*)**.** *Let G be a 2-connected graph and x_1, x_2, y_1, y_2 be vertices in G. Then the following two statements are equivalent.*

(i) *G does not contain edge-disjoint (x_1, y_1)-path and (x_2, y_2)-path.*
(ii) *G is contractible to the 4-circuit $\hat{x}_1\hat{x}_2\hat{y}_1\hat{y}_2\hat{x}_1$ or to a graph which is obtained from a 2-connected plane cubic graph by selecting a facial circuit and inserting the distinct vertices $\hat{x}_1, \hat{x}_2, \hat{y}_1, \hat{y}_2$ in that cyclic order on edges of that circuit.*

Problem 1.3 is also called integer k-commodity flow problem which was studied recently by Seymour [17].

In this paper, similar to [15] and [18], we will provide a complete characterization for graphs with or without the multiple weak 2-linkage property. It is obvious that Problem 1.3 is stronger than Problem 1.2.

Later in this paper, we will apply our characterization in the study of integer flow problems for signed graphs (see Theorem 1.7).

The following is one of our main theorems.

Theorem 1.5. *Let G be a 2-connected graph and $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ $(k \geq 2)$ be vertices in G. Then the following are equivalent.*

(i) *For any $i \neq j$, G does not contain edge-disjoint (x_i, y_i)-path and (x_j, y_j)-path.*
(ii) *The graph G can be contracted to the $2k$-circuit C_1 on the vertices $\hat{x}_1, \dots, \hat{x}_k, \hat{y}_1, \dots, \hat{y}_k$ or to a cubic graph G' which can be obtained from a 2-connected cubic plane graph by selecting a facial circuit C_2 and inserting the vertices $\hat{x}_1, \dots, \hat{x}_k, \hat{y}_1, \dots, \hat{y}_k$ on the edges of C_2 in such a way that for every pair $\{i, j\} \subseteq \{1, \dots, k\}$, the vertices $\hat{x}_i, \hat{x}_j, \hat{y}_i, \hat{y}_j$ are around C_1 or C_2 in this cyclic order.*
(iii) *Let (G^+, w) be the weighted graph where G^+ is obtained from G by adding edges $F = \{x_iy_i : i \in [1, k]\}$ and the weight $w : E(G^+) \to \{0, 1\}$ such that $w(e) = 0$ if $e \in E(G)$ and $w(e) = 1$ if $e \in F$. Then G^+ contains no pair of edge-disjoint circuits with odd weight.*

Note that (ii) is equivalent to the following:

(ii) *There is a permutation π on $[1, k]$ and G is contractible to the $2k$-circuit $\hat{z}_1\hat{z}_2 \dots \hat{z}_{2k}\hat{z}_1$ or to a graph obtained from a 2-connected plane cubic graph by selecting a facial circuit and inserting the $2k$ distinct vertices $\hat{z}_1, \hat{z}_2, \dots, \hat{z}_{2k}$ in that cyclic order on edges of that circuit, where $\{z_i, z_{k+i}\} = \{x_{\pi(i)}, y_{\pi(i)}\}$ for each $i \in [1, k]$.*

Theorem 1.5 can be applied to characterize signed graphs without edge-disjoint unbalanced circuits.

Theorem 1.6. *Let G be a 2-connected signed graph with negativeness $\epsilon(G) = k \geq 2$ and with $|N(G)| = \epsilon(G)$, where $N(G) = \{x_1y_1, x_2y_2, \dots, x_ky_k\}$ is the set of all negative edges of G. Then the following are equivalent.*

(i) *G contains no edge-disjoint unbalanced circuits.*
(ii) *$G - N(G)$ is contractible to a 2-connected graph containing no edge-disjoint $(\hat{x}_i, \hat{y}_i)$-path and $(\hat{x}_j, \hat{y}_j)$-path for any $i \neq j$.*

(iii) *The graph G can be contracted to a cubic graph G' such that either $G' - \{\hat{x}_1\hat{y}_1, \ldots, \hat{x}_k\hat{y}_k\}$ is a $2k$-circuit C_1 on the vertices $\hat{x}_1, \ldots, \hat{x}_k, \hat{y}_1, \ldots, \hat{y}_k$ or can be obtained from a 2-connected cubic plane graph by selecting a facial circuit C_2 and inserting the vertices $\hat{x}_1, \ldots, \hat{x}_k, \hat{y}_1, \ldots, \hat{y}_k$ on the edges of C_2 in such a way that for every pair $\{i, j\} \subseteq \{1, \ldots, k\}$, the vertices $\hat{x}_i, \hat{x}_j, \hat{y}_i, \hat{y}_j$ are around the circuit C_1 or C_2 in this cyclic order.*

The characterization of signed graphs without edge-disjoint unbalanced circuits can be further applied to study integer flows of this family of signed graphs.

In 1983, Bouchet [3] proposed a flow conjecture which states that *every flow-admissible signed graph admits a nowhere-zero* 6*-flow*. This conjecture remains open. Bouchet [3] himself proved that such signed graphs admit nowhere-zero 216-flows. Zýka [26] further proved that such signed graphs admit nowhere-zero 30-flows. DeVos [4] improved Zýka's result to 12-flows.

In this paper, we confirm Bouchet's conjecture for signed graphs containing no edge-disjoint unbalanced circuits by applying Theorem 1.6.

Theorem 1.7. *Every flow-admissible signed graph without edge-disjoint unbalanced circuits admits a nowhere-zero* 6*-flow.*

2. Notation and terminology

For terminology and notations not defined here we follow [2,5,20]. Graphs or signed graphs considered in this paper are finite and may have multiple edges or loops.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For $U \subseteq V$, denote $\delta(U) = \delta_G(U)$ the set of edges with one end in U and the other in $V \setminus U$. If $U = \{u\}$, we write $\delta(\{u\})$ for $\delta(u)$. A d-vertex is a vertex with degree d. For an edge e of a graph G, *contracting* e is done by deleting e and then (if e is not a loop) identifying its ends. For $S \subseteq E(G)$, we use G/S to denote the resulting graph obtained from G by contracting all edges in S.

For $U_1, U_2 \subseteq V(G)$, a (U_1, U_2)-path is a path which starts at a vertex in U_1 and ends at a vertex in U_2, and whose internal vertices belong to neither U_1 nor U_2; if G_1 and G_2 are subgraphs of G, we write (G_1, G_2)-path instead of $(V(G_1), V(G_2))$-path. Let $C = v_1 \ldots v_r v_1$ be a circuit. A *segment* of C is the path $v_i v_{i+1} \ldots v_{j-1} v_j \pmod r$ contained in C and is denoted by $v_i C v_j$ or $v_j C^- v_i$.

For a plane graph G embedded in the plane Π, a *face* of G is a connected topological region (an open set) of $\Pi \setminus G$. If the boundary of a face is a circuit of G, it is called a *facial circuit* of G. The facial circuit bounding the infinite face is called the *outer circuit*.

A *block* of G is a maximal subgraph that does not contain a cut-vertex. The *block tree* $B(G)$ of G is a bipartite graph with bipartition $(\mathcal{B}, S)$, where $\mathcal{B}$ is the set of blocks of G, S is the set of cut vertices of G, and a block B and a cut vertex v are adjacent in $B(G)$ if and only if B contains v. Note that $B(G)$ is a tree. If G is not 2-connected, the blocks of G which correspond to leaves of its block tree are called *leaf blocks*. An *internal vertex* of a block of G is a vertex which is not a cut vertex of G.

3. Signed graphs and flows

3.1. Signed graphs, switching operation, and orientations

A *signed graph* (G, σ) is a graph G together with a *signature* $\sigma : E(G) \to \{\pm 1\}$. If no confusion arises, we write (G, σ) as G and the signature σ will be specified only when needed. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* otherwise. For a subgraph H of G, denote $N(H)$ the set of negative edges in H. A circuit C of G is *balanced* if $|N(C)|$ is even, and is *unbalanced* otherwise. A signed graph is *balanced* if it does not contain an unbalanced circuit. A *barbell* is the union of two disjoint unbalanced circuits joined by a path that meets the circuits only at its ends.

For a signed graph G, *switching* at a vertex u means reversing the signs of all edges incident with u. Obviously, the parity of the number of negative edges in a circuit is an invariant under the switching operation. Let $\mathcal{X}_G$ be the set of signed graphs obtained from G via a sequence of switching operations. The *negativeness* of G is defined by

$$\epsilon(G) = \min\{|N(G')| : G' \in \mathcal{X}_G\}.$$

The following two propositions directly follow from the definitions of switching operation and negativeness.

Proposition 3.1. *Let G be a signed graph. Then $\epsilon(G) = |N(G)|$ if and only if $|N(G) \cap \delta(U)| \leq \frac{1}{2}|\delta(U)|$ for every $U \subseteq V(G)$.*

Proposition 3.2. *Let G be a signed graph and $\mathcal{B}$ be the set of blocks of G. Then*

$$\epsilon(G) = \sum_{B \in \mathcal{B}} \epsilon(B).$$

In a signed graph, every edge is composed of two half-edges, each of which is incident with one end. Denote the set of half-edges of G by $H(G)$ and the set of half-edges incident with v by $H(v)$. For a half-edge $h \in H(G)$, we refer to e_h as the edge containing h, and denote the other half-edge of e_h by h'. An *orientation* of G is a mapping $\tau : H(G) \to \{\pm 1\}$ such that $\tau(h)\tau(h') = -\sigma_G(e_h)$ for each $h \in H(G)$. It is convenient to think of τ as an assignment of orientations on $H(G)$. Namely, if $\tau(h) = 1$, h is a half-edge oriented away from its end and otherwise towards its end.

3.2. Flows in signed graphs

For basic definitions, properties and results about Tutte's Flow Theory, readers are referred to standard textbooks or reference books, such as, [2,5,24], etc.

Definition 3.3. Let τ be an orientation of a signed graph G and $f : E(G) \to \mathbb{Z}$ be a mapping.

(1) The boundary of f is the function $\partial f : V(G) \to \mathbb{Z}$ defined as $\partial f(v) = \sum_{h \in H(v)} \tau(h) f(e_h)$ for each vertex v.
(2) If $\partial f(v) = 0$ for each $v \in V(G)$ and $|f(e)| < k$ for each $e \in E(G)$, (τ, f) is called an integer k-flow (or simply a k-flow) of G.
(3) If $\partial f(v) \equiv 0 \pmod{k}$ for each $v \in V(G)$ and $0 \leq f(e) < k$ for each $e \in E(G)$, (τ, f) is called a modular k-flow (or simply a $\mathbb{Z}_k$-flow) of G.
(4) The support of f, denoted by $\mathrm{supp}(f)$, is the set of edges e with $f(e) \neq 0$. A flow (τ, f) is said to be nowhere-zero if $\mathrm{supp}(f) = E(G)$.

Flows on signed graphs arise naturally as duals of local tensions on non-orientable surfaces. More discussions are referred to [1,7–14,21,25].

For the sake of convenience, a nowhere-zero k-flow (resp., nowhere-zero $\mathbb{Z}_k$-flow) is abbreviated as a k-NZF (resp., a $\mathbb{Z}_k$-NZF). Observe that G admits a k-NZF (resp., a $\mathbb{Z}_k$-NZF) under an orientation τ if and only if it admits a k-NZF (resp., a $\mathbb{Z}_k$-NZF) under any orientation τ'.

A signed graph G is *flow-admissible* if it admits a k-NZF for some k. Bouchet characterized all flow-admissible signed graphs.

Proposition 3.4 (*Bouchet [3]*)**.** *A connected signed graph G is flow-admissible if and only if $\epsilon(G) \neq 1$ and there is no cut-edge b such that $G - b$ has a balanced component.*

The rest of the paper is organized as follows. Some lemmas on flows will be presented in Section 4. The proof of Theorem 1.7 will be presented in Section 5, and the proofs of Theorems 1.5 and 1.6 will be postponed to Sections 6 and 7, respectively.

4. Some lemmas on flows

In this section we present some lemmas that will be used in the proof of Theorem 1.7.

A graph is *even* if the degree of each vertex is even. Tutte proved the following result.

Proposition 4.1 (*Tutte [19]*). *A graph admits a* 2*-NZF if and only if it is even.*

The following is a straightforward observation in network theory since, for every $U \subseteq V(G)$, $\sum_{e\in\delta(U)}\phi(e) = \sum_{e\in\delta(U^c)}\phi(e)$ (by Definition 3.3).

Proposition 4.2. *If (τ, ϕ) is a positive k-NZF of a graph G, then τ is a strongly connected orientation of G.*

Let τ be an orientation of a graph G and $E_0 \subseteq E(G)$. We denote $\tau_{\widetilde{E_0}}$ the orientation of G obtained from τ by reversing the direction of every arc in E_0. Let $f : E(G) \to \mathbb{Z}_k$ be a mapping. $f_{\widetilde{E_0}}$ is the mapping of $E(G)$ defined as follows:

$$f_{\widetilde{E_0}}(e) = \begin{cases} f(e) & \text{if } e \notin E_0, \\ k - f(e) & \text{if } e \in E_0. \end{cases}$$

Lemma 4.3 (*Younger [23]*). *If a graph G admits a $\mathbb{Z}_k$-NZF (τ, f), then there is an edge subset E_0 of G such that $(\tau_{\widetilde{E_0}}, f_{\widetilde{E_0}})$ is a positive integer flow.*

DeVos [4] proved the following extension lemma on modular flows.

Lemma 4.4 (*DeVos [4]*). *Let G be a graph with an orientation τ and assume that G admits a $\mathbb{Z}_k$-NZF. If a vertex u of G has degree at most* 3 *and $\gamma : \delta(u) \to \mathbb{Z}_k \setminus \{0\}$ satisfies $\partial\gamma(u) \equiv 0 \pmod{k}$, then there is a $\mathbb{Z}_k$-NZF (τ, ϕ) of G so that $\phi|_{\delta(u)} = \gamma$, where $\phi|_{\delta(u)}$ is the restriction of ϕ on $\delta(u)$.*

We extend DeVos's lemma to integer flows in the following lemma.

Lemma 4.5. *Let G be a graph with an orientation τ and assume that G admits a k-NZF. If a vertex u of G has degree at most* 3 *and $\gamma : \delta(u) \to \{\pm 1, \ldots, \pm(k-1)\}$ satisfies $\partial\gamma(u) = 0$, then there is a k-NZF (τ, ϕ) of G so that $\phi|_{\delta(u)} = \gamma$.*

Proof. Without loss of generality, assume that $0 < \gamma(e) < k$ for each $e \in \delta(u)$. By Lemma 4.4, G has a $\mathbb{Z}_k$-NZF (τ, ϕ_1) such that $\phi_1|_{\delta(u)} = \gamma$. By Lemma 4.3, there is a subset $E_0 \subseteq E(G)$ such that $(\tau_{\widetilde{E_0}}, \phi_2)$ is a positive k-NZF where $\phi_2 = (\phi_1)_{\widetilde{E_0}}$.

If $E_0 \cap \delta(u) = \emptyset$, let $\phi_3(e) = -\phi_2(e)$ for each $e \in E_0$, and $\phi_3(e) = \phi_2(e)$ for each $e \in E(G) \setminus E_0$. Note that $\phi_3|_{\delta(u)} = \phi_1|_{\delta(u)} = \gamma$. Thus (τ, ϕ_3) is a desired k-NZF.

Now we assume $E_0 \cap \delta(u) \neq \emptyset$. Let s and t be the numbers of arcs in $E_0 \cap \delta(u)$ with their tails and their heads at u under $\tau_{\widetilde{E_0}}$, respectively. Then $s + t \geq 1$ since $E_0 \cap \delta(u) \neq \emptyset$. Note that each arc e with tail at u under $\tau_{\widetilde{E_0}}$ contributes $k - \phi_1(e)$ to $\partial\phi_2(u)$ and $-\phi(e)$ to $\partial\phi_1(u)$ since it has head at u under τ. Similarly, each arc e with head at u under $\tau_{\widetilde{E_0}}$ contributes $-(k - \phi_1(e))$ to $\partial\phi_2(u)$ and $\phi_1(e)$ to $\partial\phi_1(u)$. Since both (τ, ϕ_1) and $(\tau_{\widetilde{E_0}}, \phi_2)$ are flows of G,

$$0 = \partial\phi_2(u) = \partial\phi_1(u) + sk - tk = \partial\gamma(u) + (s-t)k = (s-t)k.$$

Since $1 \leq s + t \leq d(u) \leq 3$, we have $s = t = 1$. Let $E_0 \cap \delta(u) = \{e_1, e_2\}$, where $e_1 = uu_1$ with its tail at u and $e_2 = uu_2$ with its head at u under $\tau_{\widetilde{E_0}}$.

We first show that there is a directed circuit C containing e_1 and e_2 under $\tau_{\widetilde{E_0}}$. If $u_1 = u_2$, let C be the directed circuit consisting of e_1 and e_2. Now we assume $u_1 \neq u_2$. Since $(\tau_{\widetilde{E_0}}, \phi_2)$ is a positive integer flow, $\tau_{\widetilde{E_0}}$ is strongly connected by Proposition 4.2. Thus there is a directed path P from u_1 to u_2. Since $d(u) \leq 3$ and u_2uu_1 is a directed path from u_2 to u_1 in $\tau_{\widetilde{E_0}}$, P does not contain u. Hence uu_1Pu_2u is a directed circuit containing e_1 and e_2 under $\tau_{\widetilde{E_0}}$.

Let $E_1 = E(C)\Delta E_0$ be the symmetric difference of $E(C)$ and E_0. Then $\delta(u) \cap E_1 = \emptyset$ and $\tau_{\widetilde{E_1}} = (\tau_{\widetilde{E_0}})_{\widetilde{E(C)}}$. Let $\phi_4 : E(G) \to \mathbb{Z}$ be the mapping such that $\phi_4(e) = \phi_2(e)$ if $e \notin E(C)$ and $\phi_4(e) = k - \phi_2(e)$ if $e \in E(C)$.

That is, $(\tau_{\widetilde{E}_1}, \phi_4)$ is obtained from $(\tau_{\widetilde{E}_0}, \phi_2)$ by reversing the directions of all arcs in $E(C)$ and replacing the flow $\phi_2(e)$ with $k - \phi_2(e)$ for each arc e in C. Since C is a directed circuit and $(\tau_{\widetilde{E}_0}, \phi_2)$ is a positive k-NZF, $(\tau_{\widetilde{E}_1}, \phi_4)$ is also a positive k-NZF.

Let $\phi_5 : E(G) \to \mathbb{Z}$ be the mapping such that $\phi_5(e) = -\phi_4(e)$ if $e \in E_1$ and $\phi_5(e) = \phi_4(e)$ otherwise. Then (τ, ϕ_5) is the flow obtained from $(\tau_{\widetilde{E}_1}, \phi_4)$ by reversing the directions of all edges in E_1 and then negating their flow values. Since $\delta(u) \cap E_1 = \emptyset$, we have $\phi_5|_{\delta(u)} = \phi_4|_{\delta(u)} = (\phi_1)_{\widetilde{E}_1}|_{\delta(u)} = \gamma$. Hence (τ, ϕ_5) is a desired k-NZF. This completes the proof of Lemma 4.5. ■

To introduce and prove our second lemma, we need the following operation and several known results. Let G be a signed graph. We define the following operation.

Φ_k : add a balanced circuit or a barbell C into G if $|E(C) \setminus E(G)| \leq k$.

For a subgraph H of G, denote by $\langle H \rangle_k$ the maximum subgraph of G obtained from H via Φ_k-operations.

The following is the well-known 6-flow theorem due to Seymour.

Theorem 4.6 (*Seymour [16]*). *Every bridgeless graph admits a* 6*-NZF.*

With a similar argument to the proof of Seymour's 6-flow theorem, Zýka obtained the following result.

Lemma 4.7 (*Zýka [26]*). *Let G be a signed graph and H be a subgraph of G. If $\langle H \rangle_2 = G$, then G admits a $\mathbb{Z}_3$-flow (τ, f) such that $E(G) \setminus E(H) \subseteq supp(f)$.*

Unlike unsigned graphs, for signed graphs, admitting a $\mathbb{Z}_k$-NZF does not guarantee that the signed graph admits a k-NZF. However, the following lemma gives a sufficient condition to guarantee the existence of a 3-NZF if the signed graph admits a $\mathbb{Z}_3$-NZF.

Lemma 4.8 (*Xu and Zhang [22]*). *Let G be a bridgeless signed graph. Then G admits a* 3*-NZF if and only if G admits a $\mathbb{Z}_3$-NZF.*

The following lemma was proved for unsigned graphs originally but can be easily extended to signed graphs.

Lemma 4.9 (*See [6] and [16]*). *Let G be a graph (or a signed graph) and k_1, k_2 be two integers. If G admits a k_1-flow (τ, f_1) and a k_2-flow (τ, f_2) such that $supp(f_1) \cup supp(f_2) = E(G)$, then $(\tau, f_1 + k_1 f_2)$ is a k_1k_2-NZF of G.*

Lemma 4.10. *Let G be a cubic signed graph with $N(G) = \{u_1u_{k+1}, \ldots, u_ku_{2k}\}$ $(k \geq 2)$. If $G - N(G)$ is the $2k$-circuit $C = u_1 \ldots u_ku_{k+1} \ldots u_{2k}u_1$ or a graph obtained from a* 2*-connected plane cubic graph by selecting a facial circuit C and inserting the vertices $u_1, \ldots, u_k, u_{k+1}, \ldots, u_{2k}$ in that cyclic order on edges of C, and if every* 2*-edge-cut of $G - N(G)$ is contained in $E(C)$, then G admits a* 6*-NZF.*

Proof. Let τ be an orientation of G. Let H be an even subgraph of G containing C such that

(a) $E(H) \subseteq E(G) \setminus N(G)$,

(b) subject to (a), $\langle H \rangle_2$ is connected and $|E(\langle H \rangle_2)|$ is as large as possible.

Since H is an unsigned even subgraph of G, by Proposition 4.1 we have the following claim.

Claim 4.1. *G admits a* 2*-flow (τ, f_1) with $supp(f_1) = E(H)$.*

Claim 4.2. *$\langle H \rangle_2 = G$.*

Proof of Claim 4.2. Note that $k \geq 2$ and each negative edge in $N(G)$ is a chord of the unsigned circuit C. Thus $N(G) \subseteq E(\langle H \rangle_2)$ and the unsigned subgraph $\langle H \rangle_2 - N(G)$ is still connected since $\langle H \rangle_2$ is connected. Denote $U = V(\langle H \rangle_2)$ and $W = V(G) - U$. Then $\langle H \rangle_2 = G[U]$. To prove Claim 4.2, it is sufficient to prove $W = \emptyset$.

Suppose to the contrary $W \neq \emptyset$. Since $\langle H\rangle_2 - N(G)$ is connected, by the maximality of $|E(\langle H\rangle_2)|$, every vertex in W has at most one neighbor in U. Thus the minimum degree of $G[W]$ is at least two since G is cubic. Let L be a component of $G[W]$. If L has a bridge, choose a bridge e such that $L - e$ has a 2-edge-connected component L_0. If L is bridgeless, let $L_0 = L$. Since the minimum degree of L is at least two, L_0 is nontrivial.

Since each 2-edge-cut of G is contained in $E(C)$, we have $|\delta_G(V(L_0))| \geq 3$. Since $|\delta_L(V(L_0))| \leq 1$, there are two distinct edges xx', yy' where $x, y \in V(L_0)$ and $x', y' \in U$. Since L_0 is 2-edge-connected and nontrivial, L_0 has two edge-disjoint (x, y)-paths P_1 and P_2. Let $H' = H \cup P_1 \cup P_2$. By the definition of Φ_2-operation, $xx', yy' \in E(\langle H'\rangle_2)$. Thus $\langle H'\rangle_2$ is connected and $|E(\langle H'\rangle_2)| > |E(\langle H\rangle_2)|$. This contradicts the maximality of $|E(\langle H\rangle_2)|$ and thus completes the proof of the claim. □

By Claim 4.2 and Lemma 4.7, G admits a $\mathbb{Z}_3$-flow (τ, f_2) with $E(G) \setminus E(H) \subseteq supp(f_2)$.

Claim 4.3. *G admits a 3-flow (τ, f_3) with $supp(f_3) = supp(f_2)$.*

Proof of Claim 4.3. Let $G' = G[supp(f_2)]$. Then G' admits a $\mathbb{Z}_3$-NZF (τ', f_2), where τ' is the restriction of τ on G'. By Lemma 4.8, it is sufficient to prove that G' is bridgeless. Note that by Theorem 1.6, G does not contain two edge-disjoint unbalanced circuits, and neither does G'.

Suppose to the contrary that G' has a bridge b. Then at least one component of $G' - b$ does not contain unbalanced circuits and thus is balanced. Let Q be a balanced component of $G' - b$, and switch some vertices of Q such that all edges of Q are positive. We use G'' to denote the new signed graph obtained from G'. Since G' admits a $\mathbb{Z}_3$-NZF, so does G''. Let (τ', f_2') be a $\mathbb{Z}_3$-NZF of G''. Since all edges of Q are positive in G'',

$$|f_2'(e)| \equiv |\sum_{v \in V(Q)} \partial f_2'(v)| \equiv 0 \pmod 3,$$

a contradiction. Thus G' is bridgeless. □

By Lemma 4.9 and Claims 4.1 and 4.3, $(\tau, f_1 + 2f_2)$ is a desired 6-NZF of G. ■

5. Proof of Theorem 1.7

The aim of this section is to prove Theorem 1.7: *Every flow-admissible signed graph without edge-disjoint unbalanced circuits admits a nowhere-zero* 6*-flow.*

Note that the balance property of a circuit and the existence of a k-NZF are two invariants under switching operations. Because G is flow-admissible and does not contain edge-disjoint unbalanced circuits, by Proposition 3.4 G is bridgeless and at most one block of G is unbalanced. If each block of G is balanced, then G is balanced. By Theorem 4.6, G admits a 6-NZF. Hence we assume that G has only one unbalanced block. By Proposition 3.2, the negativeness of this unbalanced block is equal to $\epsilon(G)$ (≥ 2). By Theorem 4.6, each balanced block admits a 6-NZF. Thus we may further assume that G is 2-connected and $\epsilon(G) = |N(G)| \geq 2$.

Denote $N(G) = \{x_1y_1, \ldots, x_ky_k\}$. By Theorem 1.6, let H be a contraction of $G - N(G)$ such that

(a) H is the $2k$-circuit $C' = \hat{z}_1\hat{z}_2 \ldots \hat{z}_{2k}\hat{z}_1$ or a graph obtained from a 2-connected plane cubic graph by selecting a facial circuit C' and inserting the $2k$ vertices $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{2k}$ in that cyclic order on edges of C', where π is a permutation on $[1, k]$ and $\{z_i, z_{k+i}\} = \{x_{\pi(i)}, y_{\pi(i)}\}$ for each $i \in [1, k]$;
(b) subject to (a), $|V(H)|$ is as small as possible.

Clearly, $G_1 = H + N(G)$ is cubic and by the minimality of H, every 2-edge-cut of H (and thus G_1) is contained in $E(C')$. Let τ be an orientation of G. By Lemma 4.10, G_1 admits a 6-NZF (τ_1, f_1), where τ_1 is the restriction of τ on G_1.

Next we will prove that (τ_1, f_1) can be extended to a 6-NZF of G. If $G = G_1$, then (τ_1, f_1) is a 6-NZF of G. So we assume $G \neq G_1$.

Pick an arbitrary vertex x of H such that the subgraph B_x of G contracted into x is nontrivial (such x exists since $G \neq G_1$). Since G_1 is cubic, denote $\delta_G(V(B_x)) = \delta_{G_1}(x) = \{e_1, e_2, e_3\}$, and for $i \in [1, 3]$, let

h_i be the half edge of e_i whose end is in B_x. We add a new vertex u to $B_x + \{h_1, h_2, h_3\}$ such that u is a common end of all h_i, and denote the new graph by G_2. Then G_2 is a bridgeless unsigned graph and thus admits a 6-NZF by Theorem 4.6.

Let τ_2 be the restriction of τ on G_2 and define $\gamma(h_i) = f_1(e_i)$ for each h_i. Note that $\tau_2(h_i) = \tau_1(h_i)$ for each h_i. Since (τ_1, f_1) is a 6-NZF of G_1, we have $\partial\gamma(u) = \partial f_1(x) = 0$. By Lemma 4.5, there is a 6-NZF (τ_2, f_2) of G_2 such that $f_2|_{\delta_{G_2}(u)} = \gamma = f_1|_{\delta_{G_1}(x)}$, and thus f_1 can be extended to all edges in $E(B_x)$. By applying the same argument to every x with nontrivial B_x, one can extend (τ_1, f_1) to a 6-NZF of G. □

6. Proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5. (iii) ⇒ (i) is trivial since (i) is a special case of (iii). We only need to show that (i) ⇒ (ii) ⇒ (iii).

(i) ⇒ (ii)

We first show that (i) ⇒ (ii) by contradiction. Let k be the smallest integer such that there is a counterexample to it and choose G to be a counterexample with $|V(G)| + |E(G)|$ minimum. Then there are $2k$ vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ such that G does not contain edge-disjoint (x_i, y_i)-path and (x_j, y_j)-path for any $i \neq j$ but (ii) does not hold. Denote $\mathcal{T} = \{x_1, x_2, \ldots, x_k, y_1, y_2 \ldots, y_k\}$. Note that $k \geq 3$ by Theorem 1.4.

By the minimality of G, we have the following claim.

Claim 6.1. *If G has a subgraph B such that there is an edge $xy \in \delta_G(V(B))$ where $x \in V(B)$ such that B is a leaf block of $G - xy$, then $|V(B) \cap \{x_i, y_i : i \in [1, k]\}| \geq 2$ unless B is a K_2 in which case $d_G(x) = 2$ and $x \in \mathcal{T}$.*

Proof of Claim 6.1. Let $G_1 = G/E(B)$ if $|E(B)| \geq 2$ and $|V(B) \cap \mathcal{T}| \leq 1$ or if $|E(B)| = 1$ and $x \notin \mathcal{T}$. Since B is a leaf block of $G - xy$, G_1 remains 2-connected and satisfies (i). Thus by the minimality of G, $G/E(B)$ satisfies (ii) and so does G, a contradiction. This proves the claim. □

By the minimality of k, there is a permutation π on $[1, k-1]$ and G is contractible to a 2-connected graph H with maximum degree at most 3 satisfying

(1) $\hat{z}_1, \ldots, \hat{z}_{k-1}, \hat{z}_{k+1}, \ldots, \hat{z}_{2k-1}$ are distinct 2-vertices of H appearing in a facial circuit C_1 of H in the cyclic order, where $\{z_i, z_{k+i}\} = \{x_{\pi(i)}, y_{\pi(i)}\}$ for $i \in [1, k-1]$.

We choose such H satisfying (1) and

(2) subject to (1), $|V(C_1) \cap \{\hat{z}_i : i \in [1, 2k]\} \cap V_2(H)|$ as large as possible where $V_2(H)$ is the set of all 2-vertices in H.

Denote $\hat{\mathcal{T}}_H = \{\hat{z}_1, \ldots, \hat{z}_{k-1}, \hat{z}_{k+1}, \ldots, \hat{z}_{2k-1}\} \cup \{\hat{z}_k, \hat{z}_{2k}\}$ and let $S \subseteq E(G)$ such that $H = G/S$. Note that $|V(C_1) \cap \hat{\mathcal{T}}_H \cap V_2(H)|$ is the maximum among all contraction H of G satisfying (1).

Since G does not contain edge-disjoint (z_i, z_{i+k})-path and (z_j, z_{j+k})-path for any $i, j \in [1, k]$ and $i \neq j$, we have the following observation.

Claim 6.2. *For any distinct $i, j \in [1, k]$, H does not contain vertex-disjoint $(\hat{z}_i, \hat{z}_{k+i})$-path and $(\hat{z}_j, \hat{z}_{k+j})$-path.*

By applying Claim 6.1, for any 2-edge-cut $\{e_1, e_2\}$ of G, every component of $G - \{e_1, e_2\}$ either contains at least two vertices in $\mathcal{T}$ or is a single vertex in $\mathcal{T}$ which has degree 2 in G. Since each 2-edge-cut in H is also a 2-edge-cut in G, the following claim holds.

Claim 6.3. *If $\{e_1, e_2\}$ is a 2-edge-cut of H, then every component of $H - \{e_1, e_2\}$ either contains at least two vertices in $\hat{\mathcal{T}}_H = \{\hat{z}_i : i \in [1, 2k]\}$ or is a single vertex in $\hat{\mathcal{T}}_H$ which is a 2-vertex in H.*

Since $\hat{z}_i \in V(C_1) \cap V_2(H)$ for each $i \in [1, k-1] \cup [k+1, 2k-1]$ by (1), we only need to show the following statements:

- $\hat{z}_k \neq \hat{z}_{2k}$ and $\hat{z}_d \neq \hat{z}_i$ for any $d \in \{k, 2k\}$ and $i \in [1, k-1] \cup [k+1, 2k-1]$ (see Claim 6.4);
- $\{\hat{z}_k, \hat{z}_{2k}\} \subset V(C_1)$ (see Claim 6.5);
- $\{\hat{z}_k, \hat{z}_{2k}\} \subset V_2(H)$ (see Claim 6.6).

From the above three statements, it is not difficult to check that G satisfies (ii) and the suppressed graph of H (contracting all 2-vertices in H) is a desired cubic plane graph which is a contradiction to the minimality of G.

Claim 6.4. *$\hat{z}_k \neq \hat{z}_{2k}$ and $\hat{z}_d \neq \hat{z}_i$ for each $d \in \{k, 2k\}$ and $i \in [1, k-1] \cup [k+1, 2k-1]$.*

Proof of Claim 6.4. Obviously, $\hat{z}_k \neq \hat{z}_{2k}$. Suppose to the contrary that $\hat{z}_d = \hat{z}_i$ for some $d \in \{k, 2k\}$ and some $i \in [1, k-1] \cup [k+1, 2k-1]$. Without loss of generality, assume $\hat{z}_k = \hat{z}_{k+1}$.

Let W be the connected subgraph of G that is contracted into $\hat{z}_k$. Since $\hat{z}_{k+1}$ is a 2-vertex of H, let $\delta_G(V(W)) = \delta_H(\hat{z}_{k+1}) = \{e_1, e_2\}$, where e_1, e_2 occur on $\hat{z}_{k-1}C_1\hat{z}_{k+2}$ cyclically, and denote by h_1, h_2 the ends of e_1, e_2 on W, respectively. Note that $h_1 \neq h_2$ and $G/E(W)$ is 2-edge-connected since G is 2-connected.

Since G is 2-connected, there are two edge-disjoint paths P_1 and P_2 in G joining $\{z_1, z_{2k}\}$ to $V(W)$ (thus to $\{h_1, h_2\}$). Without loss of generality, assume that P_1 is a (z_1, h_1)-path and P_2 is a (z_{2k}, h_2)-path.

(6.4.1) *There are no edge-disjoint (h_2, z_k)-path and (h_1, z_{k+1})-path in W.*

Otherwise let P_1' and P_2' be edge-disjoint (h_1, z_{k+1})-path and (h_2, z_k)-path in W, respectively. Then the (z_{2k}, z_k)-path $P_2h_2P_2'$ is edge-disjoint from the (z_1, z_{k+1})-path $P_1h_1P_1'$ in G, a contradiction to (i). This proves (6.4.1).

(6.4.2) *W is 2-connected.*

Suppose to the contrary that W is not 2-connected. Since $\{e_1, e_2\}$ is an edge-cut of G and G is 2-connected, h_1 and h_2 belong to distinct leaf blocks of W. Since $|V(W) \cap \mathcal{T}| \leq 2$, by Claim 6.1, each leaf block is a K_2 in which case $d_G(h_i) = 2$ for each $i = 1, 2$. Thus $\{h_1, h_2\} = \{z_k, z_{k+1}\} \subset \mathcal{T}$. Let $S' = S \setminus \{e | e$ is in and is incident with or W h_1 $h_2\}$ and let $H' = G/S'$. Then H' can be obtained from H by simply splitting the vertex $\hat{z}_{k+1}$ into two adjacent 2-vertices $\hat{z}_k$ and $\hat{z}_{k+1}$. Let C_1' be the corresponding facial circuit in H' to C_1. Then H' satisfies (1) but $|V(C_1') \cap \hat{\mathcal{T}}_{H'} \cap V_2(H')| = 1 + |V(C_1) \cap \hat{\mathcal{T}}_H \cap V_2(H)|$, a contradiction to the choice of H. This proves (6.4.2).

By (6.4.1) and (6.4.2), we can apply Theorem 1.4 on W with the vertices h_2, h_1, z_k, z_{k+1}. That is, W is contractible to a graph W' which is either a 4-circuit $C' = \hat{h}_2'\hat{h}_1'\hat{z}_k'\hat{z}_{k+1}'\hat{h}_2'$ or a graph which is obtained from a 2-connected plane cubic graph by selecting the outer circuit C' and inserting the vertices $\hat{h}_2', \hat{h}_1', \hat{z}_k', \hat{z}_{k+1}'$ in that cyclic order on edges of C'. Replacing $\hat{z}_k$ in H with W' to obtain a new contraction H' of G, the facial circuit C_1' obtained from C_1 by replacing the vertex $\hat{z}_k$ by the segment $\hat{h}_1'C'\hat{z}_k'C'\hat{z}_{k+1}'C'\hat{h}_2'$. Thus H' satisfies (1) but $|V(C_1') \cap \hat{\mathcal{T}}_{H'} \cap V_2(H')| = 1 + |V(C_1) \cap \hat{\mathcal{T}}_H \cap V_2(H)|$, a contradiction to the choice of H. This contradiction completes the proof of the claim. □

Denote the facial circuit $C_1 = u_1u_2 \ldots u_\ell u_1$ and assume that $u_1 = \hat{z}_1, u_s = \hat{z}_{k-1}$, and $u_t = \hat{z}_{k+1}$ with $1 < s < t < \ell$ since $\hat{z}_1, \ldots, \hat{z}_{k-1}, \hat{z}_{k+1}, \ldots, \hat{z}_{2k-1}$ occur in C_1 in the cyclic order. Note that $\Delta(H) \leq 3$ and $d_H(u_1) = d_H(u_s) = d_H(u_t) = 2$. Let

$$a = \min\{i : \text{there is a } (\hat{z}_k, u_i)\text{-path } P \text{ in } H - E(C_1)\},$$
$$b = \max\{i : \text{there is a } (\hat{z}_k, u_i)\text{-path } P \text{ in } H - E(C_1)\}.$$

Note that if $\hat{z}_k$ is an isolated vertex of $H - E(C_1)$, then $\hat{z}_k = u_a = u_b$.

Claim 6.5. *$\{\hat{z}_i : i \in [1, 2k]\} \subseteq V(C_1)$.*

Proof of Claim 6.5. Since $\hat{z}_i \in V(C_1)$ for each $i \in [1, k-1] \cup [k+1, 2k-1]$, we only need to show $\{\hat{z}_k, \hat{z}_{2k}\} \subset V(C_1)$. Suppose to the contrary $\hat{z}_k \notin V(C_1)$ (without loss of generality). Since H is 2-connected, there are two internally vertex-disjoint $(\hat{z}_k, C_1)$-paths, and thus $a < b$. Since $\hat{z}_k \notin V(C_1)$, $d(u_a) = d(u_b) = 3$ and thus $u_i \neq \hat{z}_j$ for any $i \in \{a, b\}$ and $j \in [1, 2k]$.

Let P_i be a $(\hat{z}_k, u_i)$-path in $H - E(C_1)$ for each $i \in \{a, b\}$. Let P be a $(\hat{z}_{2k}, C_1)$-path in H and let u_μ be the other end of P on C_1. Without loss of generality, assume that u_a lies in the segment $\hat{z}_1C_1\hat{z}_2$.

(6.5.1) *$u_aC_1u_b$ contains the vertex $\hat{z}_2$.*

Suppose to the contrary that $u_aC_1u_b$ does not contain the vertex $\hat{z}_2$. Then u_b also lies in the segment $\hat{z}_1C_1\hat{z}_2$. Let $e_a = u_au_{a-1}$ and $e_b = u_bu_{b+1}$. Since C_1 is a facial circuit of H and the maximum degree of H is at most 3, $\{e_a, e_b\}$ is a 2-edge-cut of H by the definition of a and b. Note that the component of $H - \{e_a, e_b\}$ containing $\hat{z}_k$ does not contain any other $\hat{z}_i$ than $\hat{z}_k$ and has at least three vertices. This contradicts Claim 6.3 and this contradiction proves (6.5.1).

(6.5.2) Let $\{i, j\} \subset \{a, b, \mu\}$ such that $\mu \in \{i, j\}$ and $i < j$. *Then neither $u_iC_1u_j$ nor $u_jC_1u_i$ contains both $\hat{z}_d$ and $\hat{z}_{d+k}$ for any $d \in [1, k-1]$.*

Suppose to the contrary that $u_iC_1u_j$ contains both $\hat{z}_d$ and $\hat{z}_{d+k}$ for some $d \in [1, k-1]$. Without loss of generality, assume $i = a$ and $j = \mu$. Then $P_ju_jC_1u_iP_i$ is a $(\hat{z}_{2k}, \hat{z}_k)$-path which is vertex-disjoint from the $(\hat{z}_d, \hat{z}_{d+k})$-path $\hat{z}_dC_1\hat{z}_{d+k}$, a contradiction to Claim 6.2. So $u_iC_1u_j$ does not contain both $\hat{z}_d$ and $\hat{z}_{d+k}$ for any $d \in [1, k-1]$. By symmetry, we can show that $u_jC_1u_i$ does not contain both $\hat{z}_d$ and $\hat{z}_{d+k}$ for any $d \in [1, k-1]$ and thus this proves (6.5.2).

Since u_a lies in the segment of $\hat{z}_1C\hat{z}_2$, u_μ must lie in the segment of $\hat{z}_{k+1}C\hat{z}_{k+2}$ by (6.5.2). Since u_μ lies in the segment of $\hat{z}_{k+1}C\hat{z}_{k+2}$, by (6.5.2) again, u_b must lie in the segment of $\hat{z}_1C\hat{z}_2$. Thus $\hat{z}_2$ does not belong to the segment $u_aC_1u_b$, a contradiction to (6.5.1). This completes the proof of the claim. □

Claim 6.6. *$\{\hat{z}_i : i \in [1, 2k]\} \subseteq V_2(H)$. That is, $d_H(\hat{z}_k) = d_H(\hat{z}_{2k}) = 2$.*

Proof of Claim 6.6. Suppose to the contrary $d_H(\hat{z}_k) = 3$ (without loss of generality). $\hat{z}_k$ is not an isolated vertex of $H - E(C_1)$. Since G is 2-connected, $a \neq b$. Since both $\hat{z}_{k-1} = u_s$ and $\hat{z}_{k+1} = u_t$ are isolated vertices in $H - E(C_1)$, $a \neq s$ and $b \neq t$.

(6.6.1) *There is a $(\hat{z}_k, \hat{z}_{2k})$-path P and an $i \in [1, k-1]$ such that P and $\hat{z}_iC_1\hat{z}_{i+k}$ share only one common vertex which is $\hat{z}_k$.*

Without loss of generality, we assume that $\hat{z}_k$ lies in the segment $u_sC_1u_t = \hat{z}_{k-1}C_1\hat{z}_{k+1}$. Then $\hat{z}_{2k}$ must lie in the segment $\hat{z}_{2k-1}C_1\hat{z}_1$. Otherwise it is easy to find vertex-disjoint $(\hat{z}_k, \hat{z}_{2k})$-path and $(\hat{z}_i, \hat{z}_{k+i})$-path for some $i \in [1, k-1]$, a contradiction to Claim 6.2. Note that $a \neq s$ and $b \neq t$.

We first show that either $a < s$ or $b > t$. Otherwise suppose $s < a < b < t$. Let $e_a = u_au_{a-1}$ and $e_b = u_bu_{b+1}$. Note that C_1 is a facial circuit of H and the maximum degree of H is at most 3. By the definition of a and b, $\{e_a, e_b\}$ is a 2-edge-cut of H. Moreover, $\hat{z}_k$ is the unique vertex of $\{\hat{z}_i : i \in [1, 2k]\}$ in the component of $H - \{e_a, e_b\}$ containing $\hat{z}_k$. This contradicts Claim 6.3 since the component is nontrivial.

If $a < s$, let P_a be a $(\hat{z}_k, u_a)$-path in $H - E(C_1)$. Then the $(\hat{z}_k, \hat{z}_{2k})$-path $\hat{z}_kP_au_aC_1^-\hat{z}_{2k}$ and the $(\hat{z}_{k-1}, \hat{z}_{2k-1})$-path $\hat{z}_{k-1}C_1\hat{z}_{2k-1}$ only share the vertex $\hat{z}_k$.

If $b > t$, let P_b be a $(\hat{z}_k, u_b)$-path in $H - E(C_1)$. Then the $(\hat{z}_k, \hat{z}_{2k})$-path $\hat{z}_kP_bu_bC_1\hat{z}_{2k}$ and the $(\hat{z}_1, \hat{z}_{k+1})$-path $u_1C_1u_t = \hat{z}_1C_1\hat{z}_{k+1}$ only share the vertex $\hat{z}_k$. This proves (6.6.1).

Without loss of generality, we take $i = 1$ in (6.6.1). That is, there is a $(\hat{z}_k, \hat{z}_{2k})$-path P such that P and $\hat{z}_1C_1\hat{z}_{k+1}$ share only one common vertex $\hat{z}_k$.

Let W be the subgraph of G that is contracted into $\hat{z}_k$. Denote $\delta_G(V(W)) = \delta_H(\hat{z}_k) = \{e_1, e_2, e_3\}$ where e_1 is not in C_1 and e_2 and e_3 are in C_1. Then $e_1 \in E(P)$ and both e_2 and e_3 are in $\hat{z}_1C\hat{z}_{k+1}$. Denote the ends of e_1, e_2, e_3 in W by h_1, h_2, h_3, respectively.

(6.6.2) *W does not contain edge-disjoint (z_k, h_1)-path and (h_2, h_3)-path.*

If W contains edge-disjoint (z_k, h_1)-path and (h_2, h_3)-path, one can easily find edge-disjoint (z_k, z_{2k})-path and (z_1, z_{k+1})-path since the $(\hat{z}_k, \hat{z}_{2k})$-path P and $\hat{z}_1C_1\hat{z}_{k+1}$ share only one common vertex $\hat{z}_k$ in H. This proves (6.6.2).

(6.6.3) *W is 2-connected.*

Suppose to the contrary that W is not 2-connected. Then it has at least two leaf blocks and thus $|E(W)| \geq 2$. Since $|\delta_G(V(W))| = 3$, there must be a leaf block of W which is incident with exactly one of e_1, e_2 and e_3, denoted e_i. Then W contains a leaf block of $G - e_i$. By Claim 6.4, z_k is the only vertex in $\{z_i : i \in [1, 2k]\} = \mathcal{T}$ contained in W. By Claim 6.1, $d_G(h_i) = 2$ and $h_i = z_k$. Let e_i' be the edge in W incident with h_i. Then $\{e_i', e_\alpha, e_\beta\}$ is a 3-edge-cut where $\{\alpha, \beta\} = \{1, 2, 3\} \setminus \{i\}$. Let W' be the component of $G - \{e_i', e_\alpha, e_\beta\}$ contained in W. Then W' contains no z_i for each $i \in [1, 2k]$ and

$|E(W')| = |E(W)| - 1$. Moreover $G/E(W')$ remains 2-connected and satisfies (i). By the minimality of G, $G/E(W')$ satisfies (ii) and so does G, a contradiction. This proves (6.6.3).

By (6.6.2) and (6.6.3), we apply Theorem 1.4 on W with the vertices z_k, h_2, h_1, h_3. Then W is contractible to a graph W_1 which is either the 4-circuit $C' = \hat{h}_1\hat{h}_2\hat{z}'_k\hat{h}_3\hat{h}_1$ or a graph which is obtained from a 2-connected plane cubic graph by selecting the outer circuit C' and inserting the vertices $\hat{h}_1, \hat{h}_2, \hat{z}'_k, \hat{h}_3$ in that cyclic order on edges of C'. Without loss of generality, assume $\hat{h}'_2C'\hat{h}'_3$ contains $\hat{z}'_k$. Replace $\hat{z}_k$ in H with W_1 to obtain a new contraction H' of G. The facial circuit C'_1 is obtained from C_1 by replacing the vertex $\hat{z}_k$ with the segment $\hat{h}'_2C'\hat{h}'_3$ which contains $\hat{z}'_k$ as a 2-vertex. Thus H' satisfies (1) but $|V(C'_1) \cap \hat{\mathcal{T}}_{H'} \cap V_2(H')| = 1 + |V(C_1) \cap \hat{\mathcal{T}}_H \cap V_2(H)|$, a contradiction to the choice of H. This contradiction completes the proof of Claim 6.6. □

(i) ⇒ (ii) follows from Claims 6.4, 6.5, and 6.6.

(ii) ⇒ (iii)

Now we show (ii) ⇒ (iii). Note that if $k = 2$, then (ii) implies (iii). We prove by contradiction. Let G be a counterexample such that

(a) k is as small as possible;

(b) subject to (a), $|E(G)|$ is as small as possible.

Since G is a counterexample, let C_1 and C_2 be a pair of edge-disjoint circuits with odd weight in G^+. Let $F_i = F \cap E(C_i)$ where each $|F_i|$ is odd.

By (ii) (or (ii)′), there is a permutation π on $[1, k]$ and a subset $S \subseteq E(G)$ such that G/S is the $2k$-circuit $C = z_1z_2 \dots z_{2k}z_1$ or a graph obtained from a 2-connected plane cubic graph by selecting a facial circuit C and inserting the $2k$ vertices $z_1, z_2, \dots, z_{2k}$ in that cyclic order on edges of C, where $\{z_i, z_{k+i}\} = \{\hat{x}_{\pi(i)}, \hat{y}_{\pi(i)}\}$ for $i \in [1, k]$.

Claim 6.7. *$G = G/S$, $G^+ = C \cup C_1 \cup C_2$, and $F = F_1 \cup F_2$.*

Proof of Claim 6.7. We first show $G = G/S$. Clearly, $G^+/S = G/S + F$. Let X be a component of $G^+[S]$. Then X is contracted into a vertex of G^+/S. Let C_0 be a circuit of G^+. Since G^+/S is cubic, $C_0 \cap X$ is either a null graph or is a segment of C_0. Thus $C_0/E(C_0 \cap X)$ is a circuit. This implies that $C_0/(E(C_0) \cap S)$ is still a circuit after contracting each component of $G^+[S]$. Since $S \subseteq E(G)$, C_0 and $C_0/(E(C_0) \cap S)$ have the same number of edges in F. Hence $C_1/(E(C_1) \cap S)$ and $C_2/(E(C_2) \cap S)$ remain a pair of edge-disjoint circuits with odd weight in G^+/S. Therefore by the minimality of $E(G)$, $G = G/S$.

Now we show that $G^+ = C \cup C_1 \cup C_2$ and $F = F_1 \cup F_2$. It is obvious that $G^+ = C \cup C_1 \cup C_2$ implies $F = F_1 \cup F_2$. Let $G' = C \cup C_1 \cup C_2$. Then G' is a 2-edge-connected subgraph of G^+. Since G^+ is cubic and the edges in F are chords of C, $G' - F$ is 2-connected. Note that C_1 and C_2 are still a pair of edge-disjoint circuits with odd weight in G'. If G' is a proper subgraph of G^+, then $G' - F$ is a proper 2-connected subgraph of G satisfying (ii) and $(G' - F)^+ = G'$, a contradiction to the minimality of G. Therefore $G^+ = C \cup C_1 \cup C_2$ and thus $F = F_1 \cup F_2$. □

Note that C_1 and C_2 are vertex-disjoint since G^+ is cubic and C_1 is edge-disjoint from C_2. For $i = 1, 2$, let $\mathcal{P}_i$ be the set of $|F_i|$ paths which consist of $C_i - F_i$. Then all paths in $\mathcal{P}_1 \cup \mathcal{P}_2$ are pairwise vertex-disjoint. Let $P \in \mathcal{P}_1 \cup \mathcal{P}_2$. It is obvious that two ends of P both are in $\{z_1, z_2, \dots, z_{2k}\}$. We denote P by $P_{\alpha,\beta}$ if its end vertices are z_α and z_β with $\beta > \alpha$, and define the pace of $P_{\alpha,\beta}$ as

$$\min\{\beta - \alpha, 2k - (\beta - \alpha)\}.$$

Let $P_{\alpha',\beta'}$ be a path of $\mathcal{P}_1 \cup \mathcal{P}_2$ with smallest pace. Without loss of generality, assume that $P_{\alpha',\beta'} \in \mathcal{P}_1$. We further assume that $k \geq \beta' > \alpha' = 1$.

Claim 6.8. $\beta' = 2$.

Proof of Claim 6.8. Suppose to the contrary $\beta' \geq 3$. Since $G^+ = C \cup C_1 \cup C_2$ and $F = F_1 \cup F_2$ by Claim 6.7, z_2 must be contained in some path $P_{2,\beta''} \in \mathcal{P}_1 \cup \mathcal{P}_2$. Then $P_{2,\beta''}$ must cross the path $P_{1,\beta'}$ since G is plane and $P_{1,\beta'}$ has the smallest pace, which contradicts the fact that G^+ is cubic. This completes the proof of the claim. □

Claim 6.9. $z_{k+1}z_{k+2} \notin E(C_1 \cup C_2)$.

Proof of Claim 6.9. Since G^+ is cubic and C_1 passes through z_{k+1} and z_{k+2}, the edge $z_{k+1}z_{k+2} \notin E(C_2)$. If $z_{k+1}z_{k+2} \in E(C_1 \cup C_2)$, then $z_{k+1}z_{k+2} \in E(C_1)$. Therefore, $C_1 = z_1P_{1,2}z_2z_{k+2}z_{k+1}z_1$ is an even-weighted circuit, a contradiction. This contradiction completes the proof of the claim. □

Let $G^* = G^+ - \{z_1z_{k+1}, z_2z_{k+2}\} + z_{k+1}z_{k+2}$, and $F^* = F - \{z_1z_{k+1}, z_2z_{k+2}\}$. Then G^* does contain a pair of edge-disjoint odd-weighted circuits $C'_1 = C_1 - z_{k+1}z_1P_{1,2}z_2z_{k+2} + z_{k+1}z_{k+2}$ and C_2.

Since $z_1, z_2, \ldots, z_{2k}$ appear on C in this cyclic order, the segment $z_{k+1}Cz_{k+2}$ contains no vertices in $\{z_1, z_2, \ldots, z_{2k}\}$ as internal vertices. Thus the circuit obtained from C by replacing $z_{k+1}Cz_{k+2}$ with the edge $z_{k+1}z_{k+2}$ is also a facial circuit in $G^* - F^*$. Since $|F^*| = k - 2 < |F| = k$, by the minimality of k, G^* contains no edge-disjoint odd-weighted circuits, a contradiction. This contradiction completes the proof of (ii) ⇒ (iii) and thus the proof of Theorem 1.5.

7. Proof of Theorem 1.6

The aim of this section is to prove Theorem 1.6.

(ii) ⇔ (iii) follows from Theorem 1.5. We only need to show that (iii) ⇒ (i) ⇒ (ii).

(iii) ⇒ (i)

Let G be a counterexample with $|E(G)|$ minimum. Let G' be the contraction described in (iii). Then by the minimality of G, for each vertex $\hat{x} \in V(G') \setminus \{\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{2k}\}$, we have $\hat{x} = x$. By Theorem 1.5 ((ii) ⇒ (iii)), $G - N(G)$ contains some cut vertices. Let B be a leaf block of $G - N(G)$. Since G' is 2-connected, B must be contained in W where W is the subgraph of G which was contracted into $\hat{z}_i$ for some $i \in [1, 2k]$. Moreover, B is also a block of $G - \{z_iz_{i+k}\}$ and $V(B) \cap \{z_1, \ldots, z_k, z_{k+1}, \ldots, z_{2k}\} = \{z_i\}$. Thus z_i is not a cut vertex of $G - N(G)$ and $G/E(B)$ is 2-connected since G is 2-connected. This implies that every unbalanced circuit of G containing some edges of B must pass through the negative edge z_iz_{i+k}. Since G is a counterexample, let C_1 and C_2 be two edge-disjoint unbalanced circuits of G. Thus $C_1/(E(C_1) \cap E(B))$ and $C_2/(E(C_2) \cap E(B))$ are two edge-disjoint unbalanced circuits of $G/E(B)$. Therefore, $G/E(B)$ is also a counterexample but $|E(G/E(B))| < |E(G)|$, a contradiction to the minimality of G. This proves that (iii) implies (i).

(i) ⇒ (ii)

Let G be a counterexample with $|E(G)|$ minimum. By (i), there are no edge-disjoint (x_i, y_i)-path and (x_j, y_j)-path for any $1 \le i < j \le k$. Thus to obtain a contradiction it suffices to show that $G - N(G)$ is 2-connected.

Suppose to the contrary that $G - N(G)$ is not 2-connected. If there are two blocks of $G - N(G)$ such that each contains two ends of some negative edge, then it is obvious that G has two edge-disjoint unbalanced circuits, a contradiction. Since $G - N(G)$ is not 2-connected, it has at least two leaf blocks. Thus it has a leaf block B which contains at most one end of each negative edge. Let z be the unique cut vertex of $G - N(G)$ in B and let $N' \subseteq V(B)$ be the set of vertices incident with a negative edge in G. Clearly, $|N' \setminus \{z\}| > 0$ since G is 2-connected.

If $|N'| = 1$, then z is not adjacent to a negative edge and thus $G' = G/E(B)$ remains 2-connected and still satisfies (i). Note that the negativeness is an invariant under contracting some positive edges. Thus, by the minimality of G, there is $S \subseteq E(G') \setminus N(G')$ such that $(G' - N(G'))/S$ satisfies (ii). It follows that $S \cup E(B) \subseteq E(G) \setminus N(G)$ and $(G - N(G))/(S \cup E(B)) = (G' - N(G'))/S$ satisfies (ii), a contradiction.

Thus $|N'| \ge 2$ and $|V(B)| \ge 2$. Let x_1y_1 and x_2y_2 be two negative edges with $x_1, x_2 \in N'$ and $y_1, y_2 \in V(G) \setminus V(B)$. If $B = K_2 = \{e\}$, then $x_1 \ne x_2$; otherwise $\{e, x_1y_1, x_2y_2\}$ is a 3-edge-cut of G and it contains two edges in $N(G)$, a contradiction to Proposition 3.1. Since $x_1 \ne x_2$ if $B = K_2$ and B is 2-connected if $B \ne K_2$, there are edge-disjoint (x_1, z)-path P_1 and (x_2, z)-path P_2 in B.

Let $H = G - N(G) - (V(B) \setminus \{z\})$. Then H is connected since $G - N(G)$ is connected and B is a leaf block of $G - N(G)$. We now claim that H contains edge-disjoint (z, y_1)-path and (z, y_2)-path. Otherwise by Menger's theorem, there is a cut-edge e in H (and thus in $G - N(G)$) separating z from $\{y_1, y_2\}$. So $\{e, x_1y_1, x_2y_2\}$ is a 3-edge-cut of $G - N(G) + \{x_1y_1, x_2y_2\}$ and it contains two edges in $N(G)$, a contradiction to Proposition 3.1.

Let P'_1 and P'_2 be two edge-disjoint (z, y_1)-path and (z, y_2)-path in $G - N(G) - (V(B) \setminus \{z\})$, respectively. Then $x_1P_1zP'_1y_1x_1$ and $x_2P_2zP'_2y_2x_2$ are two edge-disjoint unbalanced circuits in G, a contradiction to (i). This proves that (i) implies (ii) and thus proves Theorem 1.6.

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