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Unique Fulkerson coloring of Petersen minor-free cubic graphs

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A R T I C L E I N F O

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A B S T R A C T

Let G be a cubic graph and the graph $2G$ is obtained by replacing each edge of G with a pair of parallel edges. A proper 6-edge-coloring of $2G$ is called a *Fulkerson coloring* of G. It was conjectured by Fulkerson that every bridgeless cubic graph has a Fulkerson coloring. In this paper we show that *for a Petersen-minor free Graph G, G is uniquely Fulkerson colorable if and only if G constructed from K_4 via a series of $Y - \Delta$-operations (expending a vertex by a triangle).* This theorem is a partial result to the conjecture that, *for a Petersen-minor free Graph G, G is uniquely 3-edge-colorable if and only if G constructed from K_4 via a series of $Y - \Delta$-operations (expending a vertex by a triangle).*

1. Introduction

Let G be a cubic graph and the graph $2G$ is obtained from G by replacing each edge with a pair of parallel edges. A *Fulkerson covering* is a collection of six perfect matchings that covers each edge exactly twice. Hence, a Fulkerson covering of G is a 1-factorization of $2G$.

The following conjecture is one of the major open problems in graph theory.

Conjecture 1.1 (*Berge, Fulkerson, [4]*). *Every 2-connected cubic graph has a Fulkerson covering.*

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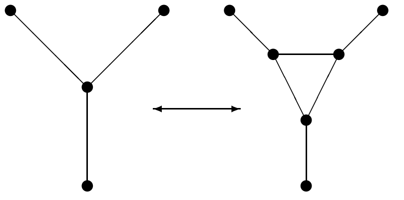

Fig. 1. $Y - \Delta$ operation.

We say a cubic graph G is *uniquely Fulkerson coverable* if G has precisely one Fulkerson covering. In this paper, we prove the following theorem.

Theorem 1.2. *Let G be a bridgeless Petersen-minor free simple cubic graph. If G is uniquely Fulkerson coverable, then G contains a triangle.*

It is not hard to see that, for 3-edge-colorable cubic graphs, the uniquely Fulkerson coverable property implies the uniquely 3-edge-colorable property. Hence, Theorem 1.2 is related to the following conjecture.

Conjecture 1.3 ([10]). *If a bridgeless Petersen-minor free simple cubic graph is uniquely 3-edge-colorable, then G must contain a triangle.*

An earlier conjecture proposed by Fiorini and Wilson [1,2] states that *every simple planar uniquely 3-edge-colorable cubic graph must contain a triangle*. This conjecture was proved by Fowler and Thomas [3] with a similar approach as the map 4-color theorem.

Theorem 1.2 is a weaker version of Conjecture 1.3 in which the condition of uniquely 3-edge-coloring is replaced with uniquely Fulkerson covering.

2. $Y - \Delta$ operation and $\langle K_4 \rangle$-graphs

Definition 2.1. The $Y - \Delta$ operation is an operation of a cubic graph that expands a vertex to a triangle or contracts a triangle to a single vertex. (See Fig. 1.) Let G be a cubic graph and denote $\langle G \rangle$ be the set of all cubic graphs which can be obtained from G by a series of $Y - \Delta$ operations.

Some small examples of $\langle K_4 \rangle$-graphs are illustrated in Fig. 2.

The following parity lemma is a well-known fact in graph theory.

Lemma 2.2. *Let G be a cubic graph with a perfect matching M. Then, for every edge-cut T of G,*

$$|T \cap M| \equiv |T| (\text{mod } 2).$$

By Lemma 2.2, we can see that the $Y - \Delta$-operation preserves the following properties of a cubic graph:

- ⋆. 3-edge-coloring;
- ⋆. Number of 3-edge-colorings;
- ⋆. Fulkerson covering;
- ⋆. Number of Fulkerson coverings; etc.

Thus, for a graph G described in Conjecture 1.3 and Theorem 1.2, if we recursively contract triangles in G ($Y - \Delta$ operations), the graph G eventually becomes K_4. That is, graphs in Conjecture 1.3 and Theorem 1.2 are characterized as $\langle K_4 \rangle$-graphs and they can be therefore restated as follows.

Conjecture 1.3′. *Let G be a bridgeless Petersen-minor free cubic graph. Then G is uniquely 3-edge-colorable if and only if $G \in \langle K_4 \rangle$.*

Theorem 1.2′. *Let G be a bridgeless Petersen-minor free cubic graph. Then G is uniquely Fulkerson coverable if and only if $G \in \langle K_4 \rangle$.*

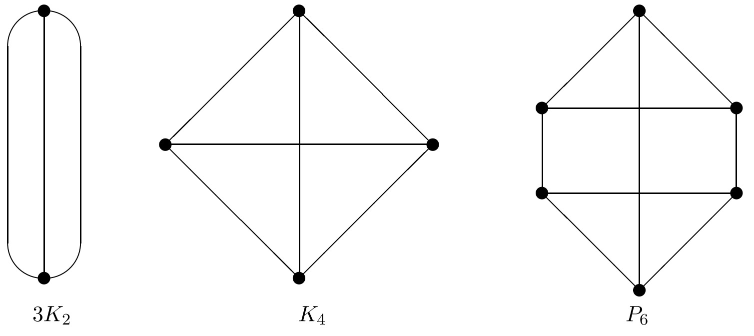

Fig. 2. Three small $\langle K_4 \rangle$-graphs.

3. Lemmas

Notice that if a cubic graph G is 3-edge-colorable, then a Fulkerson covering can be easily obtained by taking each color class twice.

Definition 3.1. A Fulkerson covering $\mathcal{F} = \{F_1, \ldots, F_6\}$ is *trivial* if there is a labeling of the index such that $F_i = F_{i+3}$ for $i = 1; 2; 3$, and non-trivial otherwise.

In [11] (also see [5]), the last author obtained a sufficient and necessary condition for Fulkerson coloring. Here, we restate the lemma (Lemma 3.3) by introducing the following definition.

Definition 3.2. An even subgraph C of a cubic graph G is called a *Fulkerson even subgraph* if C is the union of two edge-disjoint matchings M_1, M_2 such that the suppressed cubic graph $\overline{G - M_i}$ is 3-edge-colorable, for each $i = 1, 2$. Furthermore, a Fulkerson even subgraph C is *proper* if $\emptyset \neq V(C) \neq V(G)$.

Lemma 3.3 (*Zhang [11] (Lemma 3.1 in [5])*). *Let G be a cubic graph. Then G has a Fulkerson covering if and only if G has a Fulkerson even subgraph.*

Lemma 3.3 can be extended to the following lemma which will be useful in this paper.

Lemma 3.4. *Let G be a cubic graph. Then G has a proper Fulkerson even subgraph if and only if G has a non-trivial Fulkerson covering.*

Proof. "$\Rightarrow$": Let C be a proper Fulkerson even subgraph with $C = M_1 \cup M_2$, and let $G_i = \overline{G - M_i}$ for each $i = 1, 2$. For each $i = 1, 2$ let $c_i : E(G_i) \mapsto \{(i, 1), (i, 2), (i, 3)\}$ be a 3-edge-coloring, and $\hat{c}_i$ be the corresponding coloring of $G - M_i$ (by inserting degree 2 vertices back and retaining the coloring c_i). Let

$$F_{(i,j)} = \hat{c}_i^{-1}((i, j)) \triangle M_{3-i} \tag{1}$$

for each $i = 1, 2, j = 1, 2, 3$. It is easy to see that $\mathcal{F} = \{F_{(i,j)} : i = 1, 2, \ j = 1, 2, 3\}$ is a Fulkerson covering of G. By (1), we can see how $\mathcal{F}$ covers edges of G:

(a) *If $v \notin V(C)$, then for each $e \in E(v)$, the edge e is contained in precisely one of $\{F_{(1,j)} : j = 1, 2, 3\}$ and precisely one of $\{F_{(2,j')} : j' = 1, 2, 3\}$.*

(b) *If $v \in V(C)$, let $E(v) = \{e_0, e_1, e_2\}$ with $e_1 \in M_1, e_2 \in M_2$ and $e_0 \notin E(C)$. Then the edge e_0 is contained in precisely one of $\{F_{(1,j)} : j = 1, 2, 3\}$ and precisely one of $\{F_{(2,j')} : j' = 1, 2, 3\}$; the edge e_1 is contained in precisely two of $\{F_{(2,j')} : j' = 1, 2, 3\}$ and the edge e_2 is contained in precisely two of $\{F_{(1,j)} : j = 1, 2, 3\}$.*

Since $\emptyset \neq V(C) \neq V(G)$, by applying (a) to some vertex not in C and applying (b) to some vertex of C, $\mathcal{F}$ does not satisfy Definition 3.1, and therefore, is not trivial.
"$\Leftarrow$": Let $\mathcal{F} = \{F_1, \ldots, F_6\}$ be a non-trivial Fulkerson covering.

Let $v \in V(G)$ and $E(v) = \{e_1, e_2, e_3\}$. Without loss of generality, let $e_i \in F_i$ and F_{3+i} for each $i = 1, 2, 3$.

Case 1. If $F_1 \cap F_2 \neq \emptyset$. Consider

$$M_1 = \{e : e \in F_i \cap F_j : i \neq j \in \{1, 2, 3\}\}$$

and

$$M_2 = \{e : e \in F_i \cap F_j : i \neq j \in \{4, 5, 6\}\}.$$

Here $v \notin V(M_1 \cup M_2)$. Thus, $C = M_1 \cup M_2$ is a proper Fulkerson even subgraph.

Case 2. If $F_i \cap F_j = \emptyset$, for each $i, j \in \{1, 2, 3\}$. Then $\{F_1, F_2, F_3\}$ is a 1-factorization of G. So is $\{F_4, F_5, F_6\}$ since $\mathcal{F}$ is a double cover. Without loss of generality, let $F_1 \neq F_4$ as $\mathcal{F}$ is non-trivial. So, either $F_4 \cap F_2$ or $F_4 \cap F_3 \neq \emptyset$. Say, $F_4 \cap F_2 \neq \emptyset$. Let

$$M_1' = \{e : e \in F_i \cap F_j : i \neq j \in \{4, 2, 3\}\}$$

and

$$M_2' = \{e : e \in F_i \cap F_j : i \neq j \in \{1, 5, 6\}\}.$$

Here, $M_1' \neq \emptyset$ and $v \notin M_1' \cup M_2'$. That is, $C' = M_1' \cup M_2'$ is a proper Fulkerson even subgraph. ■

Lemma 3.5 (*Robertson, Sanders, Seymour and Thomas, [7,8]*)**.** *Let G be a bridgeless Petersen-minor free cubic graph. Then G is 3-edge-colorable.*

Lemma 3.6. *Let G be a bridgeless Petersen-minor free cubic graph. Then G is uniquely Fulkerson coverable if and only if G does not have a non-trivial Fulkerson covering.*

Proof. "$\Rightarrow$": By Lemma 3.5, G is 3-edge-colorable with a 3-edge coloring c. Let $\mathcal{F}_c$ be the trivial Fulkerson covering induced by the 3-edge-coloring c. This Fulkerson covering must be unique since G is uniquely Fulkerson coverable. So G does not have a non-trivial Fulkerson covering.
"$\Leftarrow$": Prove by contradiction. First, G has a trivial Fulkerson covering by Lemma 3.5. Now, assume that G has more than one Fulkerson covering. Let $\mathcal{F}^i = \{F_1^i, \ldots, F_6^i\}$ be two distinct trivial Fulkerson coverings for $i = 1, 2$ where $F_j^i = F_{j+3}^i$ for $j = 1, 2, 3$ (by Definition 3.1).

Then we have two distinct 1-factorizations of G, which are $\{F_1^i, F_2^i, F_3^i\}$ for $i = 1, 2$. A non-trivial Fulkerson covering can be derived as $\{F_j^i : i = 1, 2, j = 1, 2, 3\}$ which contradicts to G does not have a non-trivial Fulkerson covering. ■

4. Hamiltonian weight and unique Fulkerson covering

Due to the nature of Fulkerson covering, it is closely related to the problems of cycle covers. In this section, we will show some relationship between hamiltonian weight and unique Fulkerson covering.

Definition 4.1. A weight $w : E(G) \mapsto \{1, 2\}$ is called a $(1, 2)$-*weight* of G, denoted by (G, w). A family $\mathcal{F}$ of circuits (cycles) of G is called *a faithful circuit (cycle) cover* of (G, w) if each edge e is contained in precisely $w(e)$ members of $\mathcal{F}$.

Observed that the total weight of each edge-cut must be even in order to have a faithful cycle cover for (G, w). So we say a weight is *eulerian* if the total weight of each edge-cut is even.

However, a $(1, 2)$-eulerian weight of a bridgeless graph does not always guarantee a faithful cycle cover. The Petersen graph P_{10} together with a $(1, 2)$-eulerian weight w_{10} does not have a faithful cycle cover, where $w_{10}(e) = 2$ if $e \in M$ for a perfect matching M of P_{10} and $w_{10}(e) = 1$ otherwise.

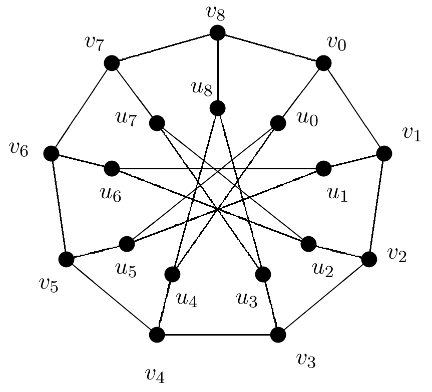

Fig. 3. Generalized Petersen graph $P(9, 2)$.

Definition 4.2. Let w be a $(1, 2)$-eulerian weight of a cubic graph G. A faithful cycle cover $\mathcal{F}$ of (G, w) is *hamiltonian* if $\mathcal{F}$ is a set of two Hamilton circuits. A $(1, 2)$-eulerian weight w of G is *hamiltonian* if every faithful circuit cover of (G, w) is hamiltonian.

Notice that if G is a uniquely 3-edge-colorable cubic graph with the 1-factorization $\mathcal{M} = \{M_1, M_2, M_3\}$ and let w be a $(1, 2)$-eulerian weight of G where $w(e) = 2$ if $e \in M_3$ and $w(e) = 1$ otherwise, then w has a hamiltonian cover $\{M_1 \bigcup M_3, M_2 \bigcup M_3\}$.

We notice that the properties of "*uniquely* 3-*edge-coloring*" and "*hamiltonian weight*" are not the same. The generalized Petersen graph $P(9, 2)$ (discovered by Tutte, see Fig. 3) is uniquely 3-edge-colorable [9], but does not admit a hamiltonian weight [10].

The following lemma shows some relationship between unique Fulkerson covering and hamiltonian weight.

Lemma 4.3. *Let G be a Petersen-minor free cubic graph. If G is uniquely Fulkerson coverable, then G admits a hamiltonian weight.*

Proof. By Lemma 3.5, the graph G is 3-edge-colorable with a 1-factorization $\{F_1, F_2, F_3\}$. Since G is uniquely Fulkerson coverable, the unique Fulkerson covering of G is the trivial Fulkerson covering derived from the 1-factorization.

Define a $(1, 2)$-eulerian weight $w : E(G) \mapsto \{1, 2\}$ where $w(e) = 2$ if $e \in F_1$ and $w(e) = 1$ otherwise. Then (G, w) has a faithful cycle cover $\{F_1 \bigcup F_2, F_1 \bigcup F_3\}$.

Prove by contradiction: We assume that w is not a hamiltonian weight of G. Let $\mathcal{C}$ be a faithful circuit cover of G with respect to w that some member $C \in \mathcal{C}$ is not a Hamilton circuit.

Since C is alternatively in F_1 and $F_2 \cup F_3$, let $M_1 = C \cap F_1$ and $M_2 = C - F_1$.

We first claim that *both $G - M_i$ are bridgeless for $i = 1, 2$* (before we apply Lemmas 3.5 and 3.4).

In $G - M_1$, $F_2 \cup F_3$ remains as a Hamilton circuit, hence, it is bridgeless. In $G - M_2$, $\mathcal{C} - \{C\}$ remains as a cycle cover of $G - M_2$. Thus, every edge of $G - M_2$ is covered by some circuit of $\mathcal{C} - \{C\}$ and therefore, $G - M_2$ is bridgeless.

By Lemma 3.5, both $G - M_i$ are 3-edge-colorable for $i = 1, 2$. Thus, by Definition 3.2, the circuit C is a proper Fulkerson even subgraph. Further applying Lemma 3.4, G has a non-trivial Fulkerson covering $\mathcal{F}'$. This contradicts Lemma 3.6 that G is unique Fulkerson coverable. ■

In Lemmas 3.6 and 4.3, the condition of G being "Petersen-minor free" is needed, as we notice that the Petersen graph P_{10} itself has a unique Fulkerson covering which consists of all six perfect matchings, and it does not admit a hamiltonian weight. To finish this section, we present the following lemma which will be used in the proof of the main theorem.

Lemma 4.4 (*Lai and Zhang [6]*)**.** *Let G be a 3-connected Petersen-minor free cubic graph. Then G admits a hamiltonian weight if and only if $G \in \langle K_4 \rangle$.*

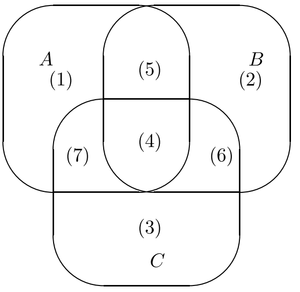

Fig. 4. Relationship between theorem and conjectures.

5. Proof of Theorem 1.2′

The "⇐" part is immediate from the property of $Y - \Delta$ operation.

The "⇒" part: First we show that if G is uniquely Fulkerson coverable, then G is 3-connected: Proof by contradiction, assume that G has a 2-edge-cut $T = \{e_1, e_2\}$. Let $\{F_1, F_2, F_3\}$ be the unique 1-factoring of G, by Lemma 2.2 we may assume that $T \subset F_1$. Then $F_2 \bigcup F_3$ is an even cycle but not a Hamilton circuit, denote one of the even circuits of $F_2 \bigcup F_3$ as C_T. We can derive a new 1-factoring $\{F_1, F_2 \Delta C_T, F_2 \Delta C_T\}$ which contradicts to the unique Fulkerson covering of G. Thus G is 3-edge-connected, hence G is 3-connected as G is cubic.

By Lemma 4.3, the graph G admits a hamiltonian weight. By Lemma 4.4, $G \in \langle K_4 \rangle$. This completes the proof.

6. Discussion

We notice that: The Petersen graph has a unique Fulkerson covering which consists of all six perfect matchings. And any graph obtained from the Petersen graph via a series of $Y - \Delta$ operations also has a unique Fulkerson covering. So it is natural to ask if we can extend Theorem 1.2 with the requirement of Petersen-minor free substituted by some other relaxed conditions, or find a complete characterization of all uniquely Fulkerson coverable graphs.

The following is an observation that the generalized Petersen graph $P(9, 2)$ (see Fig. 3) is uniquely 3-edge colorable, but is not uniquely Fulkerson coverable.

First, $P(9, 2)$ is uniquely 3-edge colorable, so a trivial Fulkerson covering $\mathcal{F}$ can be immediately obtained from the unique 3-edge coloring.

Next, we have a proper Fulkerson even subgraph $C = M_1 \bigcup M_2$ where

$$M_1 = \{v_7u_7, u_3u_8, u_4v_4, v_5v_6\}, \qquad M_2 = \{u_7u_3, u_8u_4, v_4v_5, v_6v_7\}.$$

Notice that $\overline{G - M_1}$ has a Hamilton circuit $v_0v_1v_2v_3u_2u_6u_3u_5u_0v_8v_0$ and $\overline{G - M_2}$ also has a Hamilton circuit $v_0u_0v_3v_2v_1u_1u_5u_6u_2v_8v_0$. So by Lemma 3.4, G has a non-trivial Fulkerson covering $\mathcal{F}'$. Thus there are at least two different Fulkerson coverings of $P(9, 2)$.

Here we propose the following conjectures.

Conjecture 6.1. *Let G be a uniquely 3-edge-colorable cubic graph, if G is uniquely Fulkerson coverable, then G must contain a triangle.*

Conjecture 6.2. *If G be a uniquely Fulkerson coverable cubic graph, then $G \in \langle K_4, P_{10} \rangle$.*

Now, we denote

- ⋆. the set A be the cubic graphs which are Petersen-minor free,
- ⋆. the set B be the cubic graphs which are uniquely Fulkerson coverable
- ⋆. and the set C be the cubic graph which are uniquely 3-edge colorable. (See Fig. 4.)

The following are our analysis and characterization.

(1) $A - (B \bigcup C) \neq \emptyset$ since $K_{3,3} \in A - (B \bigcup C)$.
(2) $B - (A \bigcup C) \neq \emptyset$ since $P_{10} \in B - (A \bigcup C)$.
(3) $C - (A \bigcup B) \neq \emptyset$ since $P(9, 2) \in C - (A \bigcup B)$.
(4) $A \bigcap B \bigcap C = \langle K_4 \rangle$, as in Theorem 1.2, we showed that $A \bigcap B = \langle K_4 \rangle$ and also $A \bigcap B \bigcap C = \langle K_4 \rangle$.
(5) $A \bigcap B - C = \emptyset$, by Theorem 1.2.
(6) In Conjecture 6.1, it is conjectured that $B \bigcap C - A = \emptyset$.
(7) In Conjecture 6.2, it is conjectured that $C \bigcap A - B = \emptyset$.

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