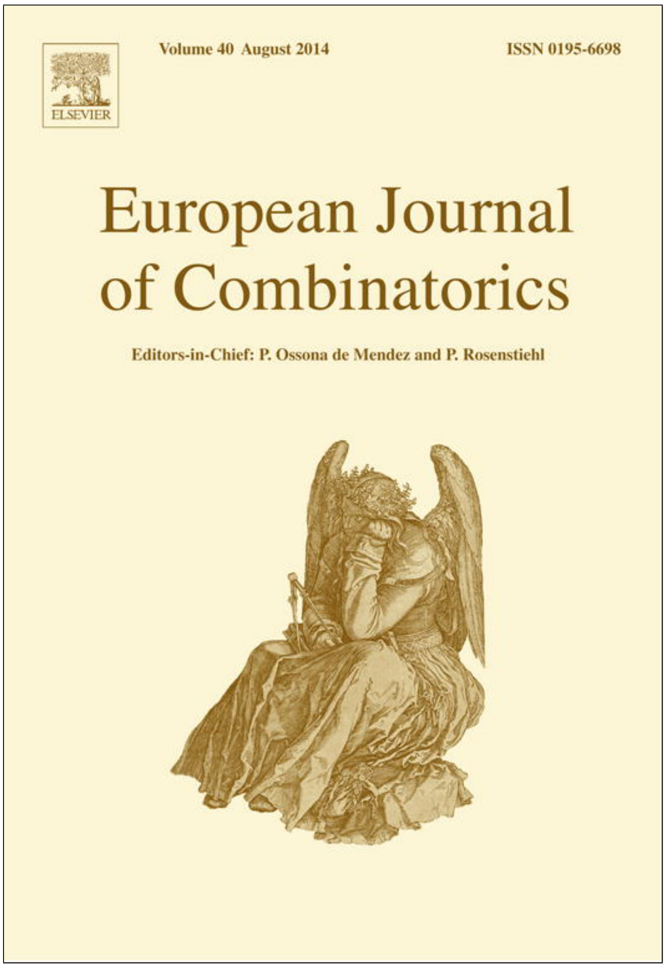
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A note on an extremal problem for group-connectivity

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ABSTRACT

In Luo et al. (2012), an extremal graph theory problem is proposed for group connectivity: for an abelian group A with $|A| \geq 3$ and an integer $n \geq 3$, find $ex(n, A)$, where $ex(n, A)$ is the maximum number so that every n-vertex simple graph with at most $ex(n, A)$ edges is not A-connected. In this paper, we determine the values $ex(n, A)$ for $A = Z_k$ where $k \geq 3$ is an odd integer and for $A = Z_4$, each of which solves some open problem proposed in Luo et al. (2012). Furthermore, the values $ex(n, Z_4)$ also imply a characterization of Z_4-connected graphic sequences.

1. Introduction

The *vertex set* and *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. An *orientation* D of a graph G is a directed graph by assigning a direction to each edge in $E(G)$. For a direct graph D and a vertex $v \in V(D)$, we use $E^+_D(v)$ (or $E^-_D(v)$, respectively) to denote the set of edges with tails (or heads, respectively) at v and we denote $d^+_D(v) = |E^+_D(v)|$ and $d^-_D(v) = |E^-_D(v)|$ the outdegree and indegree of v respectively. Let A be an abelian group. The *order* of A is denoted by $|A|$. The degree of the vertex $v \in V(G)$ is the number of edges incident with it, denoted by $d_G(v)$ (or simply $d(v)$).

A mapping $\beta : V(G) \to A$ is an A-boundary if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{k}$ where $k = |A|$. A graph G is A-connected, if for every A-boundary β, there exists an orientation D and a mapping $f : E(D) \to$

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$A \setminus \{0\}$ so that $f^+(v) - f^-(v) \equiv \beta(v) \pmod{k}$ for each vertex $v \in V(G)$ where $f^+(v) = \sum_{e \in E_D^+(v)} f(e)$ and $f^-(v) = \sum_{e \in E_D^-(v)} f(e)$. In particular, if $\beta(v) = 0$ for every vertex v, such pair (D, f) is called a *nowhere-zero A-flow*.

Note that, whether a bridgeless G admits a nowhere-zero A-flow only depends on the order of A. Tutte [12] proved that a graph G admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for any abelian group A with $|A| = k$.

Unlike the group flow, it is unknown whether the structure of the group A plays any role in A-connectivity. In fact, it is an open problem to determine if any Z_4-connected graph is also $Z_2 \times Z_2$-connected, or vise versa proposed in [2].

The concept of A-connectivity was introduced by Jaeger, Linial, Payan, and Tarsi [2] as a generalization of nowhere-zero flows. A-connected graphs are contractible configurations of A-flow and play an important role in the study of group flows and integer flows.

Major open problems in this area are Tutte's celebrated 3-, 4-, and 5-flow conjectures and group Z_3-, Z_5-connectivity conjectures by Jaeger, Linial, Payan, and Tarsi [2]. Readers are referred to [13] for in-depth accounts and [6,11] for recent results.

A sparse graph may still admit a nowhere-zero A-flow even for $|A| = 2, 3, 4$ such as any cycle admits a nowhere-zero Z_2-flow while it is not A-connected if $|A|$ is not big enough. It has been observed (see [5,7,14]) that higher density of edges in a graph would imply smaller group connectivity and that graphs with small group connectivity number cannot have too few edges. The following extremal problem on group connectivity was studied in [8]: for an abelian group A with $|A| \geq 3$ and an integer $n \geq 3$, find $ex(n, A)$, where $ex(n, A)$ is the maximum number so that every n-vertex simple graph with at most $ex(n, A)$ edges is not A-connected. Lai et al. also asked a similar question (see Problem 7.21 of [4]). The following result was proved in [8].

Theorem 1.1 ([8]). *Let A be an abelian group with $|A| = k$.*

(1) $3n/2 \leq ex(n, Z_3) \leq 2n - 3$ *if* $n \geq 6$.
(2) *If* $|A| = k \geq 4$ *and* $n \geq k$, *then* $ex(n, A) \leq \lceil \frac{(n-1)(k-1)}{k-2} \rceil - 1$.

As observed in [8], $ex(n, A) = n - 1$ if $3 \leq n < |A|$, $ex(3, Z_3) = 3$, $ex(4, Z_3) = 6$, and $ex(5, Z_3) = 7$.

It is conjectured in [8] that those upper bounds stated in Theorem 1.1 above are the exact values for $ex(n, A)$.

Conjecture 1.2 ([8]). $ex(n, A) = \lceil \frac{(n-1)(k-1)}{k-2} \rceil - 1$ *if* $|A| \geq 4$ *and* $n \geq |A|$ *or if* $A = Z_3$ *and* $n \geq 6$.

When $|A| = 4$, a little more general result was proved (stated as Theorem 1.3 below), which concludes that any simple graph G with minimum degree at least 2 and with at least $ex(n, A) + 1$ edges either is A-connected or there is another A-connected simple graph H with the same degree sequence as G. A sequence of n nonnegative integers is graphic if it is the degree sequence of a simple graph and such a graph is called a *realization* of the degree sequence.

Theorem 1.3 ([8]). *Let A be an abelian group with $|A| = 4$, $n \geq 3$ be an integer, and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with minimum degree at least 2. If the degree sum $d_1 + d_2 + \cdots + d_n \geq 3n - 3$, then π has a realization that is A-connected.*

The following conjecture is also proposed in [8].

Conjecture 1.4 ([8]). *Let A be an abelian group with $|A| = 4$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with minimum degree at least 2. Then π has an A-connected realization if and only if the degree sum $d_1 + d_2 + \cdots + d_n \geq 3n - 3$.*

It has been extensively studied whether a degree sequence has a realization with certain properties. A noticeable application (see [10]) of graph realization with 4-flows has been found in the design of critical partial Latin squares which leads to the proof of the so-called simultaneous edge-coloring conjecture by Keedwell [3] and Cameron [1]. All graphic sequences which have realizations admitting a nowhere-zero 3-flow or 4-flow are characterized in [9,10] respectively.

In this paper, we prove the following results.

Theorem 1.5. (1) *$ex(n, Z_k) = \lceil \frac{(n-1)(k-1)}{k-2} \rceil - 1$ if $k \geq 5$ is odd and $n \geq k$ or if $k = 3$ and $n \geq 6$.*
(2) $ex(n, Z_4) = \lceil \frac{3n-3}{2} \rceil - 1$ if $n \geq 4$.

Our results confirm Conjecture 1.2 for $A = Z_k$ where $k \geq 3$ is an odd integer and for $A = Z_4$. Theorem 1.5(2) together with Theorem 1.3 also implies that Conjecture 1.4 is true for Z_4, which characterizes Z_4-connected graphic sequences.

Theorem 1.6. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with minimum degree at least 2. Then π has a Z_4-connected realization if and only if $d_1 + d_2 + \cdots + d_n \geq 3n - 3$.*

Proof. The proof simply follows from Theorems 1.3 and 1.5(2). ■

2. Proof of Theorem 1.5

The proofs of (1) and (2) of Theorem 1.5 are different. In this section, we will prove them separately.

2.1. $ex(n, Z_k)$ where k is odd

By Theorem 1.1, we only need to prove the following.

Theorem 2.1. *Let $k \geq 3$ be an odd integer. Every simple Z_k-connected graph with $n \geq k$ vertices has at least $\frac{(n-1)(k-1)}{k-2}$ edges.*

Proof. Let G be a Z_k-connected graph with n vertices. Denote $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Let $t = s = k - 2$. Define a Z_k-boundary of G, β as

$$\beta(v_i) \equiv t - sd(v_i) \pmod{k} \quad (1)$$

for each $i = 1, \ldots, n-1$ and $\beta(v_n) \equiv -\sum_{i=1}^{n-1} \beta(v_i) \pmod{k}$. Since G is Z_k-connected, there is a pair (D_1, f_1) such that $f_1(e) \in Z_k \setminus \{0\}$ for each edge e and $f_1^+(v) - f_1^-(v) \equiv \beta(v) \pmod{k}$ for each vertex v. Since k is odd, for each integer i, either i or $i-k$ is odd and $i \equiv i - k \pmod{k}$. If $f_1(e)$ is even, we can obtain an equivalent β-flow of G by reversing the direction of e and replacing $f_1(e)$ with $k - f_1(e)$. Thus we may further assume $f_1(e) \in \{1, 3, \ldots, (k-2)\}$.

In the rest of the proof, we regard (D_1, f_1) as an integer-valued (not in Z_k any more) flow of G (maybe unbalanced) such that $f_1(e) \in \{1, 3, \ldots, (k-2)\}$ for each edge $e \in D_1$ and $f_1^+(v) - f_1^-(v) \equiv \beta(v) \pmod{k}$ for each vertex $v \in V(G)$.

Let D be the directed graph obtained from D_1 by adding a new directed edge corresponding to each edge in D_1 with an opposite direction. We define an integer-valued function $f : E(D) \mapsto \mathbf{Z}$ as $f(uv) = f_1(uv)$ if $uv \in D_1$ and $f(uv) = -f_1(uv)$ if $uv \in D - D_1$. Then it is easy to check that (D, f) satisfies the following properties:

(1) $f^+(v) = f_1^+(v) - f_1^-(v) \equiv \beta(v) \pmod{k}$ for each vertex v;

(2) $f(uv) + f(vu) = 0$ for any two edges uv and vu in D;

(3) $\sum_{e \in E(D)} f(e) = 0$;

(4) $f(e) \in \{\pm 1, \pm 3, \ldots, \pm(k-2)\}$ and $f(e)$ is odd for each edge e in $E(D)$;

(5) $|E(D)| = 2|E(G)|$.

Define another integer-valued function $g : E(D) \mapsto \mathbf{Z}$ such that for each edge $e \in E(D)$, $g(e) = s + f(e)$. Hence, we have the following properties.

(i) For each edge $e \in E(D)$, $g(e) \geq 0$ and is even.

Let $e \in E(D)$. By (4), $f(e)$ is odd and $-(k-2) \leq f(e) \leq k-2$. Since $s = k-2$, we have $g(e) = s + f(e) = k - 2 + f(e) \geq k - 2 + (-(k-2)) = 0$. Since k is odd, $g(e) = k - 2 + f(e)$ is even.

(ii) For each $i = 1, 2, \ldots, n$, $g^+(v_i) = \sum_{e \in E_D^+(v_i)} g(e) \geq 0$ and is even.

(ii) follows from (i) directly because $g(e) \geq 0$ and is even for each edge $e \in D$.

(iii) $g^+(v_i) \geq 2k - 2$ for each $1 \leq i \leq n - 1$.

Let $1 \le i \le n-1$. Since $g^+(v_i) = \sum_{e \in E_D^+(v_i)} g(e) = sd(v_i) + f^+(v_i)$, by (1) we have

$$g^+(v_i) = sd(v_i) + f^+(v_i) \equiv sd(v_i) + \beta(v_i) \pmod{k}.$$

Since $\beta(v_i) \equiv t - sd(v_i) \pmod{k}$ by the definition of β (see Eq. (1)) and since $t = k-2$, we further have

$$g^+(v_i) = sd(v_i) + f^+(v_i) \equiv sd(v_i) + \beta(v_i) \equiv t \equiv k-2 \pmod{k}.$$

Since $g^+(v_i) \ge 0$ by (ii), $g^+(v_i) \ge k-2$. Since $g^+(v_i)$ is even by (i) and $k-2$ is odd, we have $g^+(v_i) \ge k-2+k = 2k-2$.

Since $|E(D)| = 2|E(G)|$ and $g(e) = s + f(e)$ and since $\sum_{e \in E(D)} f(e) = 0$ by (3), we have

$$\sum_{u \in V(G)} g^+(u) = \sum_{u \in V(G)} \sum_{e \in E_D^+(u)} g(e) = \sum_{e \in E(D)} g(e) = 2s|E(G)| + \sum_{e \in E(D)} f(e) = 2(k-2)|E(G)|.$$

On the other hand since $g^+(u) \ge 0$ for each vertex u by (ii) and $g^+(v_i) \ge 2k-2$ for each $i = 1, \ldots, n-1$ by (iii), we have

$$\sum_{u \in V(G)} g^+(u) \ge \sum_{i=1}^{n-1} g^+(v_i) \ge (2k-2)(n-1).$$

So $|E(G)| \ge \frac{(n-1)(k-1)}{k-2}$. ■

2.2. The values of $ex(n, Z_4)$

By Theorem 1.1, we only need to prove the following result.

Theorem 2.2. *Every simple Z_4-connected graph with $n \ge 4$ vertices has at least $\frac{3n-3}{2}$ edges.*

Proof. Let G be Z_4-connected with n vertices. Denote $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Define a Z_4-boundary $\beta : V(G) \mapsto Z_4$ as $\beta(v_i) \equiv d(v_i) - 1 \pmod{4}$, $1 \le i \le n-1$ and $\beta(v_n) \equiv -\sum_{i=1}^{n-1} \beta(v_i) \pmod{4}$. Since G is Z_4-connected, there is a pair (D, f) so that $f(e) \in Z_4 \setminus \{0\}$ for each edge e in D and for each vertex v in G

$$f^+(v) - f^-(v) \equiv \beta(v) \pmod{4}.$$

Since $2 \equiv -2$ and $3 \equiv -1$ in Z_4, we may assume $f(e) \in \{1, 2\}$ for each edge e in D.

Let D_1 be the subgraph of D consisting of edges with weight 1 and let E_2 denote the set of edges with weight 2.

Claim. *For each vertex $v \in \{v_1, \ldots, v_{n-1}\}$, $d_{D_1}^+(v) - d_{D_1}^-(v) \le d(v) - 3$.*

Proof of Cliam. Let $v \in \{v_1, \ldots, v_{n-1}\}$ and $a = |[E_D^+(v) \cup E_D^-(v)] \cap E_2|$. Then $d_{D_1}^+(v) + d_{D_1}^-(v) + a = d(v)$. Since $2 \equiv -2 \pmod{4}$, we have

$$f^+(v) - f^-(v) \equiv d_{D_1}^+(v) - d_{D_1}^-(v) + 2a \equiv \beta(v) \equiv d(v) - 1 \pmod{4}. \tag{2}$$

Since $d_{D_1}^+(v) + d_{D_1}^-(v) \equiv d_{D_1}^+(v) - d_{D_1}^-(v) \pmod{2}$, by Eq. (2), we have

$$d(v) - a = d_{D_1}^+(v) + d_{D_1}^-(v) \equiv d_{D_1}^+(v) - d_{D_1}^-(v) \equiv d(v) - 1 \pmod{2}.$$

Therefore a is odd and of course $a \ge 1$.

If $d_{D_1}^-(v) \ge 1$, then $d_{D_1}^+(v) - d_{D_1}^-(v) = d(v) - a - 2d_{D_1}^-(v) \le d(v) - 3$.

If $d_{D_1}^-(v) = 0$, then $d_{D_1}^+(v) + a = d(v)$. By Eq. (2), we have $a \equiv -1 \pmod{4}$. Hence $a \ge 3$. Therefore $d_{D_1}^+(v) - d_{D_1}^-(v) = d(v) - a \le d(v) - 3$. This completes the proof of the claim. □

Since $\sum_{i=1}^{n}(d^+_{D_1}(v_i) - d^-_{D_1}(v_i)) = 0$, $\sum_{i=1}^{n-1}(d^+_{D_1}(v_i) - d^-_{D_1}(v_i)) = d^-_{D_1}(v_n) - d^+_{D_1}(v_n)$. By the above claim, we have

$$\sum_{i=1}^{n-1}(d(v_i) - 3) \geq \sum_{i=1}^{n-1}(d^+_{D_1}(v_i) - d^-_{D_1}(v_i)) = d^-_{D_1}(v_n) - d^+_{D_1}(v_n) \geq -d(v_n).$$

Therefore, $2|E(G)| - 3n = \sum_{i=1}^{n}(d(v_i) - 3) = \sum_{i=1}^{n-1}(d(v_i) - 3) + d(v_n) - 3 \geq -d(v_n) + d(v_n) - 3 = -3$. This implies $|E(G)| \geq \frac{3n-3}{2}$. ■

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