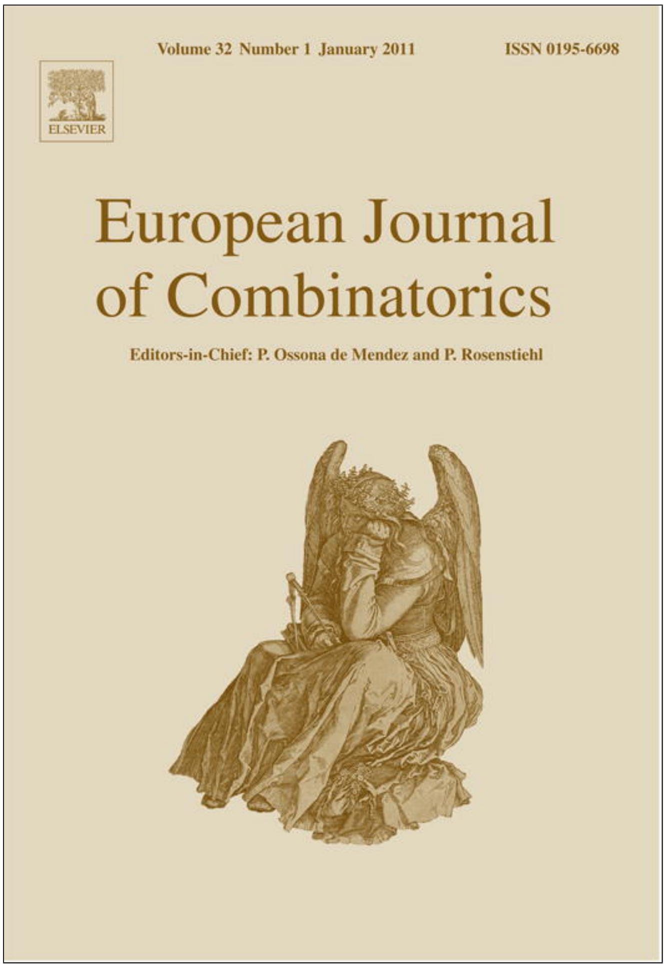
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Cycle double covers and the semi-Kotzig frame

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ABSTRACT

Let H be a cubic graph admitting a 3-edge-coloring $c : E(H) \to \mathbb{Z}_3$ such that the edges colored with 0 and $\mu \in \{1, 2\}$ induce a Hamilton circuit of H and the edges colored with 1 and 2 induce a 2-factor F. The graph H is semi-Kotzig if switching colors of edges in any even subgraph of F yields a new 3-edge-coloring of H having the same property as c. A spanning subgraph H of a cubic graph G is called a *semi-Kotzig frame* if the contracted graph G/H is even and every non-circuit component of H is a subdivision of a semi-Kotzig graph.

In this paper, we show that a cubic graph G has a circuit double cover if it has a semi-Kotzig frame with at most one non-circuit component. Our result generalizes some results of Goddyn [L.A. Goddyn, Cycle covers of graphs, Ph.D. Thesis, University of Waterloo, 1988], and Häggkvist and Markström [R. Häggkvist, K. Markström, Cycle double covers and spanning minors I, J. Combin. Theory Ser. B 96 (2006) 183–206].

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *circuit* of G is a connected 2-regular subgraph. A subgraph of G is *even* if every vertex is of even degree. An even subgraph of G is also called a *cycle* in the literature dealing with cycle covers of graphs [14,13,21]. Every even graph has a circuit decomposition. A set $\mathscr{C}$ of even subgraphs of G is *an even-subgraph double cover* (cycle double cover) if each edge of G is contained by precisely two even subgraphs in $\mathscr{C}$. The Circuit Double-Cover Conjecture was made independently by Szekeres [17] and Seymour [16].

Conjecture 1.1 (*Szekeres [17] and Seymour [16]*). *Every bridgeless graph G has a circuit double cover.*

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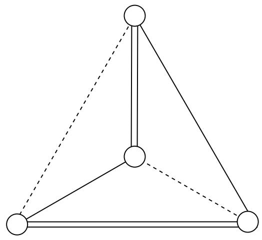

Fig. 1. The Kotzig graph K_4.

It suffices to show that the Circuit Double-Cover Conjecture holds for bridgeless cubic graphs [14]. The Circuit Double-Cover Conjecture has been verified for several classes of graphs; for example, cubic graphs with Hamilton paths [19] (also see [5]), cubic graphs with oddness 2 [11] and oddness 4 [10,8], and Petersen-minor-free graphs [1].

A cubic graph H is a *spanning minor* of a cubic graph G if some subdivision of H is a spanning subgraph of G. In [4], Goddyn showed that a cubic graph G has a circuit double cover if it contains the Petersen graph as a spanning minor. Goddyn's result was further improved by Häggkvist and Markström [7] who showed that a cubic graph G has a circuit double cover if it contains a 2-connected simple cubic graph with no more than 10 vertices as a spanning minor.

A *Kotzig graph* [15] is a cubic graph H with a 3-edge-coloring $c : E(G) \to \mathbb{Z}_3$ such that $c^{-1}(\alpha) \cup c^{-1}(\beta)$ induces a Hamilton circuit of G for every pair $\alpha, \beta \in \mathbb{Z}_3$. The family of all Kotzig graphs is denoted by $\mathcal{K}$ (see Fig. 1).

Theorem 1.2 (*Goddyn [4], Häggkvist and Markström [6]*)**.** *If a cubic graph G contains a Kotzig graph as a spanning minor, then G has a* 6*-even-subgraph double cover.*

By Theorem 1.2, any cubic graph G containing some member of $\mathcal{K}$ as a spanning minor has a circuit double cover. However, we do not know yet whether every 3-connected cubic graph contains a member of $\mathcal{K}$ as a spanning minor (Conjecture 1.3).

According to their observations [6,7], Häggkvist and Markström conjectured that every 3-connected cubic graph contains a Kotzig graph as a spanning minor. In [9], Hoffmann-Ostenhof found a counterexample for this conjecture and he suggested a modified version as follows.

Conjecture 1.3 (*Häggkvist and Markström [6], Hoffmann-Ostenhof [9]*)**.** *Every cyclical* 4*-edge-connected cubic graph contains a Kotzig graph as a spanning minor.*

Häggkvist and Markström [6] proposed another conjecture (Conjecture 2.3) in a more general form. We will discuss this conjecture in the last section (in the remark).

One of approaches to the CDC conjecture is to find a sup-family $\mathcal{X}$ of $\mathcal{K}$ such that every bridgeless cubic graph containing a member of $\mathcal{X}$ as a spanning minor has a CDC. Following this direction of approach, Goddyn [4] and Häggkvist and Markström [6] introduce some sup-families of $\mathcal{K}$, named iterated-Kotzig graphs, switchable-CDC graphs and semi-Kotzig graphs. They will be defined in following subsections and their relations are shown in Fig. 2.

Iterated-Kotzig graphs

Definition 1.4. An *iterated-Kotzig graph H* is a cubic graph constructed as follows [6]: Let $\mathcal{K}_0$ be a set of Kotzig graphs with a 3-edge-coloring $c : E(G) \to \mathbb{Z}_3$; a cubic graph $H \in \mathcal{K}_{i+1}$ can be constructed from a graph $H_i \in \mathcal{K}_i$ and a graph $H_0 \in \mathcal{K}_0$ by deleting one edge colored with 0 from each of them and joining the two vertices of degree 2 in H_0 to the two vertices of degree 2 in H_i, respectively (the two new edges will be colored with 0; see Fig. 3).

Theorem 1.5 (*Häggkvist and Markström [6]*)**.** *If a cubic graph G contains an iterated-Kotzig graph as a spanning minor, then G has a* 6*-even-subgraph double cover.*

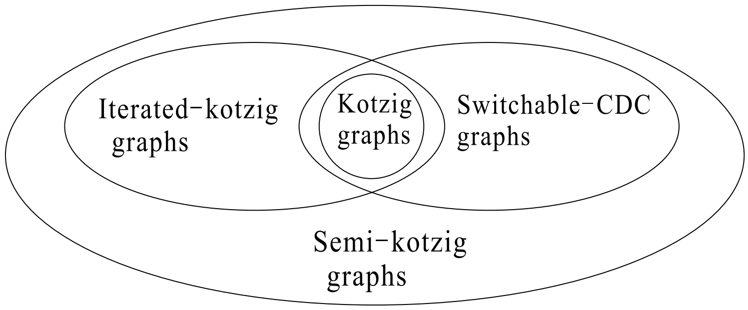

Fig. 2. The inclusion relations for these four families: Kotzig graphs, iterated-Kotzig graphs, switchable-CDC graphs, semi-Kotzig graphs.

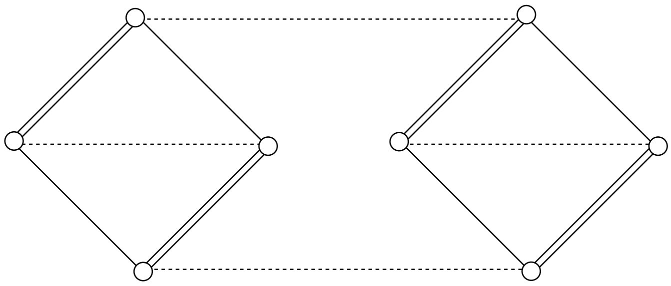

Fig. 3. An iterated-Kotzig graph generated from two K_4's.

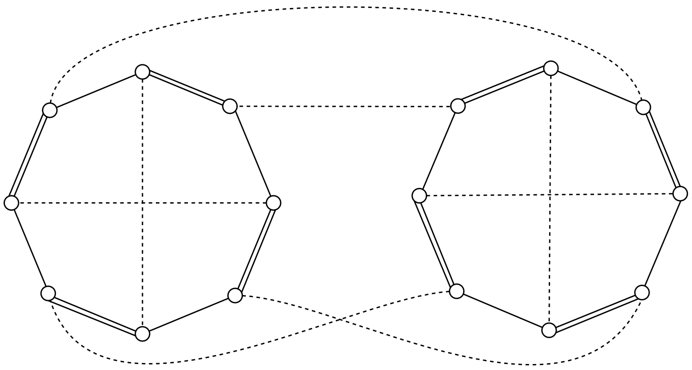

Fig. 4. A semi-Kotzig graph.

Semi-Kotzig graphs and switchable-CDC graphs

Definition 1.6. Let G be a cubic graph with a 3-edge-coloring $c : E(G) \to \mathbb{Z}_3$ and the following property:

($*$) edges in colors 0 and μ ($\mu \in \{1, 2\}$) induce a Hamilton circuit.

Let F be the even 2-factor induced by edges in colors 1 and 2. If, for every even subgraph $S \subseteq F$, switching colors 1 and 2 of the edges of S yields a new 3-edge-coloring having the property ($*$), then each of these 2^{t-1} 3-edge-colorings is called a *semi-Kotzig coloring* where t is the number of components of F. A cubic graph G with a semi-Kotzig coloring is called a *semi-Kotzig graph*. If F has at most two components ($t \leq 2$), then G is said to be a *switchable-CDC graph* (defined in [6]).

Theorem 1.7 (*Häggkvist and Markström [6]*). *If a cubic graph G contains a switchable-CDC graph as a spanning minor, then G has a 6-even-subgraph double cover.*

An iterated-Kotzig graph has a semi-Kotzig coloring and hence is a semi-Kotzig graph. But a semi-Kotzig graph is not necessary an iterated-Kotzig graph. For example, the semi-Kotzig graph in Fig. 4 is not an iterated-Kotzig graph. Hence we have the following relations (also see Fig. 2).

$$\text{Kotzig} \subset \text{Iterated-Kotzig} \subset \text{Semi-Kotzig}; \tag{1}$$

$$\text{Kotzig} \subset \text{Switchable-CDC} \subset \text{Semi-Kotzig}. \tag{2}$$

The following theorem was announced in [4] with an outline of proof.

Theorem 1.8 (*Goddyn [4]*)**.** *If a cubic graph G contains a semi-Kotzig graph as a spanning minor, then G has a* 6*-even-subgraph double cover.*

The main theorem (Theorem 1.17) of the paper strengthens all those early results (Theorems 1.2, 1.5, 1.7 and 1.8).

Kotzig frame; semi-Kotzig frame

A 2-factor F of a cubic graph is *even* if every component of F is of even length. If a cubic graph G has an even 2-factor, then the graph G has many nice properties: *G is 3-edge-colorable, G has a circuit double cover and strong circuit double cover, etc.*

The following concepts were introduced in [6] as a generalization of even 2-factors.

Definition 1.9. Let G be a cubic graph. A spanning subgraph H of G is called a *frame* of G if the contracted graph G/H is an even graph.

An alternative definition of a frame can be found in [6].

For a subgraph H of G, the *suppressed graph* $\overline{H}$ of H is the graph obtained from H by suppressing all degree 2 vertices.

Definition 1.10. Let G be a cubic graph. A frame H of G is called a *Kotzig frame* (or *iterated-Kotzig frame*, or *switchable-CDC frame*, or *semi-Kotzig frame*) of G if, for each non-circuit component H_i of H, the suppressed graph $\overline{H_i}$ is a Kotzig graph (or an iterated-Kotzig graph, or a switchable-CDC graph, or a semi-Kotzig graph, respectively).

We have, similar to the relations described in (1) and (2), the relations between those frames:

$$\text{Kotzig frame} \subset \text{Iterated-Kotzig frame} \subset \text{semi-Kotzig frame};$$

$$\text{Kotzig frame} \subset \text{Switchable-CDC frame} \subset \text{semi-Kotzig frame}.$$

Theorem 1.11 (*Häggkvist and Markström [6]*)**.** *Let G be a bridgeless cubic graph G. If G contains a Kotzig frame with at most one non-circuit component, then G has a* 6*-even-subgraph double cover.*

According to their observations, they further make the following conjecture.

Conjecture 1.12 (*Häggkvist and Markström [6]*)**.** *Every bridgeless cubic graph with a Kotzig frame has a* 6*-even-subgraph double cover.*

The following theorem provides a partial solution to Conjecture 1.12.

Theorem 1.13 (*Zhang and Zhang [22]*)**.** *Let G be a bridgeless cubic graph. If G contains a Kotzig frame H such that G/H is a tree if parallel edges are identified as a single edge, then G has a* 6*-even-subgraph double cover.*

We conjecture that the result in Conjecture 1.12 still holds if a Kotzig frame is replaced by a semi-Kotzig frame.

Conjecture 1.14. *Every bridgeless cubic graph with a semi-Kotzig frame has a* 6*-even-subgraph double cover.*

Häggkvist and Markström showed that Conjecture 1.14 holds for iterated-Kotzig frames and switchable-CDC frames with at most one non-circuit component.

Theorem 1.15 (*Häggkvist and Markström [6]*)**.** *Let G be a bridgeless cubic graph G. If G contains an iterated-Kotzig frame with at most one non-circuit component, then G has a* 6*-even-subgraph double cover.*

Theorem 1.16 (*Häggkvist and Markström [6]*)**.** *Let G be a bridgeless cubic graph G. If G contains a switchable-CDC frame with at most one non-circuit component, then G has a* 6*-even-subgraph double cover.*

The following theorem is the main result of the paper, which verifies that Conjecture 1.14 holds if a semi-Kotzig frame has at most one non-circuit component. Since Kotzig graphs and iterated-Kotzig graphs are semi-Kotzig graphs but not vice versa, Theorems 1.2, 1.5, 1.7, 1.8, 1.11, 1.15 and 1.16 are corollaries of our result. The proof of the theorem will be given in Section 2.

Theorem 1.17. *Let G be a bridgeless cubic graph. If G contains a semi-Kotzig frame H with at most one non-circuit component, then G has a* 6*-even-subgraph double cover.*

2. Proof of Theorem 1.17

The following well-known fact will be applied in the proof of the main theorem (Theorem 1.17).

Lemma 2.1. *If a cubic graph has an even* 2*-factor F, then G has a* 3*-even-subgraph double cover* $\mathscr{C}$ *such that* $F \in \mathscr{C}$.

Definition 2.2. Let H be a bridgeless subgraph of a cubic graph G. A mapping $c : E(H) \to \mathbb{Z}_3$ is called a *parity 3-edge-coloring* of H if, for each vertex $v \in H$ and each $\mu \in \mathbb{Z}_3$,

$$|c^{-1}(\mu) \cap E(v)| \equiv |E(v) \cap E(H)| \pmod 2.$$

It is obvious that if H itself is cubic, then a parity 3-edge-coloring is a proper 3-edge-coloring (traditional definition).

Preparation of the proof. Let H_0 be the component of H such that H_0 is a subdivision of a semi-Kotzig graph and each H_i, $1 \le i \le t$, be a circuit component of H of even length. Let $M = E(G) - E(H)$, and $H^* = H - H_0$.

Given an initial semi-Kotzig coloring $c_0 : E(\overline{H}_0) \to \mathbb{Z}_3$ of $\overline{H}_0$, then $F_0 = c_0^{-1}(1) \cup c_0^{-1}(2)$ is a 2-factor of $\overline{H}_0$ and $c_0^{-1}(0) \cup c_0^{-1}(\mu)$ is a Hamilton circuit of $\overline{H}_0$ for each $\mu \in \{1, 2\}$.

The semi-Kotzig coloring c_0 of $\overline{H}_0$ can be considered as an edge-coloring of H_0: each induced path is colored with the same color as its corresponding edge in $\overline{H}_0$ (note that this edge-coloring of H_0 is a parity 3-edge-coloring, which may not be a proper 3-edge-coloring).

The strategy of the proof is to show that G can be covered by three subgraphs $G(0, 1)$, $G(0, 2)$ and $G(1, 2)$ such that each $G(\alpha, \beta)$ has a 2-even-subgraph cover which covers the edges of $M \cap E(G(\alpha, \beta))$ twice and the edges of $E(H) \cap E(G(\alpha, \beta))$ once. In order to prove this, we are going to show that the three subgraphs $G(\alpha, \beta)$ have the following properties:

(i) the suppressed cubic graph $\overline{G(\alpha, \beta)}$ is 3-edge-colorable (so Lemma 2.1 can be applied to each of them);
(ii) $c_0^{-1}(\alpha) \cup c_0^{-1}(\beta) \subseteq G(\alpha, \beta)$ for each pair $\alpha, \beta \in \mathbb{Z}_3$;
(iii) the even subgraph H^* has a decomposition, into H_1^* and H_2^*, each of which is an even subgraph (here, for technical reasons, let $H_0^* = \emptyset$), such that $H_\alpha^* \cup H_\beta^* \subseteq G(\alpha, \beta)$ for each $\{\alpha, \beta\} \subset \mathbb{Z}_3$;
(iv) each $e \in M = E(G) - E(H)$ is contained in precisely one member of $\{G(0, 1), G(0, 2), G(1, 2)\}$;
(v) and most importantly, the subgraph $c_0^{-1}(\alpha) \cup c_0^{-1}(\beta) \cup H_\alpha^* \cup H_\beta^*$ in $G(\alpha, \beta)$ corresponds to an even 2-factor of $\overline{G(\alpha, \beta)}$.

Can we decompose H^* and find a partition of $M = E(G) - E(H)$ to satisfy (v)? One may also note that the initial semi-Kotzig coloring c may not be appropriate. However, the color-switchability of the semi-Kotzig component H_0 may help us to achieve the goal. The properties described above in the strategy will be proved in the following claim.

We claim that G has the following property:

(∗) *There is a semi-Kotzig coloring c_0 of $\overline{H}_0$, a decomposition $\{H_1^*, H_2^*\}$ of H^* and a partition $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$ of M such that, letting $C_{(\alpha,\beta)} = c_0^{-1}(\alpha) \cup c_0^{-1}(\beta)$,*
  (1) *for each $\mu \in \{1, 2\}$, $C_{(0,\mu)} \cup H_\mu^*$ corresponds to an even 2-factor of $\overline{G(0,\mu)} = \overline{G[C_{(0,\mu)} \cup H_\mu^* \cup N_{(0,\mu)}]}$, and*
  (2) *$C_{(1,2)} \cup H^*$ corresponds to an even 2-factor of $\overline{G(1,2)} = \overline{G[C_{(1,2)} \cup H^* \cup N_{(1,2)}]}$.*

Proof of (∗). Let G be a minimum counterexample to (∗). Let $c : E(H) \to \mathbb{Z}_3$ be a parity 3-edge-coloring of H such that

(1) the restriction of c on $\overline{H_0}$ is a semi-Kotzig coloring, and
(2) $E(H^*) \subseteq c^{-1}(1) \cup c^{-1}(2)$ (a set of mono-colored circuits).

Let

$$F = c^{-1}(1) \cup c^{-1}(2) = E(H) - c^{-1}(0).$$

Partition the matching M as follows. For each edge $e = xy \in M$, $xy \in M_{(\alpha,\beta)}$ ($\alpha \leq \beta$ and $\alpha, \beta \in \mathbb{Z}_3$) if x is incident with two α-colored edges and y is incident with two β-colored edges. So, the matching M is partitioned into six subsets:

$$M_{(0,0)}, M_{(0,1)}, M_{(0,2)}, M_{(1,1)}, M_{(1,2)} \text{ and } M_{(2,2)}.$$

Note that this partition will be adjusted whenever the parity 3-edge-coloring is adjusted.

Claim 1. *$M_{(0,\mu)} \cap G[V(H_0)] = \emptyset$ for each $\mu \in \mathbb{Z}_3$.*

Suppose that $e = xy \in M_{(0,\mu)}$ where x is incident with two 0-colored edges of H_0. Then, in the graph $\overline{G-e}$, the spanning subgraph H retains the same property as it has in G. Since $\overline{G-e}$ is smaller than G, $\overline{G-e}$ satisfies (∗): $\overline{H}_0$ has a semi-Kotzig coloring c_0 and $M - e$ has a partition $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$, and also H^* has a decomposition $\{H_1^*, H_2^*\}$. In the semi-Kotzig coloring c_0, without loss of generality, assume that y subdivides a 1-colored edge of $\overline{H}_0$. For the graph G, add e into $N_{(0,1)}$. This revised partition $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$ of M and the resulting subgraphs $G(\alpha, \beta)$ satisfy (∗). This contradicts that G is a counterexample.

Since $c^{-1}(0) \subseteq H_0$ (each component of $H - H_0 = H^*$ is mono-colored with 1 or 2), for every edge $e \in M_{(0,\mu)}$ ($\mu \in \{1, 2\}$), by Claim 1, the edge e has one endvertex incident with two 0-colored edges of H_0 and another of its endvertices belongs to $V(H - H_0) = V(H^*)$. That is,

$$M_{(0,0)} = \emptyset, \quad \text{and} \quad M_{(0,1)} \cup M_{(0,2)} \subseteq E(H_0, H^*).$$

Let

$$G' = \overline{G - M_{(0,1)} - M_{(0,2)}}.$$

Then $E(G'/F) \subseteq M_{(1,1)} \cup M_{(1,2)} \cup M_{(2,2)}$.

Claim 2. *The graph G'/F is acyclic.*

Suppose to the contrary that G'/F contains a circuit Q (including loops). In the graph $\overline{G - E(Q)}$, the spanning subgraph H remains a semi-Kotzig frame.

Then the smaller graph $\overline{G - E(Q)}$ satisfies (∗): $\overline{H}_0$ has a semi-Kotzig coloring c_0, and $M - E(Q)$ has a partition $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$, and also H^* has a decomposition $\{H_1^*, H_2^*\}$. So add all edges of $E(Q)$ into $N_{(1,2)}$. This revised partition $\{N_{(0,1)}, N_{(0,2)}, N_{(1,2)}\}$ of M and its resulting subgraphs $G(\alpha, \beta)$ also satisfy (∗) since $C_{(1,2)} \cup H^*$ corresponds to an even 2-factor of $\overline{G(1,2)} = \overline{G[C_{(1,2)} \cup H^* \cup N_{(1,2)}]}$. This is a contradiction. So Claim 2 follows.

By Claim 2, each component T of G'/F is a tree. Along the tree T, we can modify the parity 3-edge-coloring c of H as follows:

(∗∗) properly switch colors for some circuits in F so that every edge of T is incident with four same colored edges.

Note that Rule (∗∗) is feasible by Claim 2 since G'/F is acyclic. Furthermore, under the modified parity 3-edge-coloring c, $M_{(1,2)} = \emptyset$. So

$$M = M_{(0,1)} \cup M_{(0,2)} \cup M_{(1,1)} \cup M_{(2,2)}.$$

The colors of all H_i's ($i \geq 1$) give a decomposition $\{H_1^*, H_2^*\}$ of H^* where H_μ^* consists of all circuits of H^* mono-colored with μ for $\mu = 1$ and 2.

Let

$$G'' = G/H,$$

where $E(G'') = M$. Then G'' is even since H is a frame. For a vertex w of G'' corresponding to a component H_i with $i \geq 1$, there is a $\mu \in \{1, 2\}$ such that all edges incident with w belong to $M_{(0,\mu)} \cup M_{(\mu,\mu)}$. Define

$$N_{(0,\mu)} = M_{(0,\mu)} \cup M_{(\mu,\mu)}$$

for each $\mu \in \{1, 2\}$, and

$$N_{(1,2)} = M_{(1,2)} = \emptyset.$$

Hence, a vertex of G'' corresponding to H_i with $i \geq 1$ either has degree in $G''[N_{(0,\mu)}]$ the same as its degree in G'' or has degree 0 (by Rule (∗∗)). So every vertex of $G''[N_{(0,\mu)}]$ which is different from the vertex corresponding to H_0 has even degree. Since every graph has an even number of odd-degree vertices, it follows that $G''[N_{(0,\mu)}]$ is an even subgraph.

For each $\mu \in \{1, 2\}$, let $G(0, \mu) = N_{(0,\mu)} \cup (c^{-1}(0) \cup c^{-1}(\mu))$. Since $G''[N_{(0,\mu)}]$ is an even subgraph of G'', the even subgraph $c^{-1}(0) \cup c^{-1}(\mu)$ corresponds to an even 2-factor of $G(0, \mu)$. And let $G(1, 2) = F = c^{-1}(1) \cup c^{-1}(2)$ (here, $N_{(1,2)} = \emptyset$). So G has the property (∗), a contradiction. This completes the proof of (∗). □

Proof of Theorem 1.17. Let G be a graph with a semi-Kotzig frame. Then G satisfies (∗) and therefore is covered by three subgraphs $G(\alpha, \beta)(\alpha, \beta \in \mathbb{Z}_3$ and $\alpha < \beta)$ as stated in (∗).

Applying Lemma 2.1 to the three graphs $\overline{G(\alpha, \beta)}$, each $G(0, \mu)$ has a 2-even-subgraph cover $\mathscr{C}_{(0,\mu)}$ which covers the edges of $C_{(0,\mu)} \cup H_\mu^*$ once and the edges in $N_{(0,\mu)}$ twice, and $G(1, 2)$ has a 2-even-subgraph cover $\mathscr{C}_{(1,2)}$ which covers the edges of $C_{(1,2)} \cup H^*$ once and the edges in $N_{(1,2)}$ twice. So $\bigcup \mathscr{C}_{(\alpha,\beta)}$ is a 6-even-subgraph double cover of G. This completes the proof. □

Remark. In [6], Häggkvist and Markström proposed another conjecture which strengthens Theorems 1.2, 1.5 and 1.8 as follows.

Conjecture 2.3 (*Häggkvist and Markström [6]*)**.** *If a cubic bridgeless graph contains a connected* 3*-edge-colorable cubic graph as a spanning minor, then G has a* 6*-even-subgraph double cover*

In fact, Conjecture 2.3 is equivalent to every bridgeless cubic graph having a 6-even-subgraph double cover. It can be shown that the condition in Conjecture 2.3 is true for all cyclically 4-edge-connected cubic graphs.

Consider a cyclically 4-edge-connected cubic graph G, since a smallest counterexample to the 6-even-subgraph double-cover problem is cyclically 4-edge-connected and cubic. By the Matching Polytope Theorem of Edmonds [3], G has a 2-factor F such that G/F is 4-edge-connected. By the Tutte and Nash-Williams theorem [18,20], G/F contains two edge-disjoint spanning trees T_1 and T_2. By a theorem of Itai and Rodeh [12], T_1 contains a parity subgraph P of G/F. After suppressing all degree 2 vertices of $G - P$, the graph $\overline{G - P}$ is 3-edge-colorable and connected since $G/F - P$ is even and $T_2 \subset G/F - P$. So every cyclically 4-edge-connected cubic graph does contain a connected 3-edge-colorable cubic graph as a spanning minor.

Remark. In [2], Cutler and Häggkvist proved that if a cubic graph G contains a frame which has two components, one of them is a subdivision of a Kotzig graph and the other is a subdivision of a semi-Kotzig graph, then G has a cycle double cover.

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