European Journal of Combinatorics 32 (2011) 21–27

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Vertex-coloring 2-edge-weighting of graphs☆

Hongliang Lu [a], Qinglin Yu [b,1], Cun-Quan Zhang [c]

[a] *Department of Mathematics, Xi'an Jiaotong University, Xi'an, China*

[b] *Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, BC, Canada*

[c] *Department of Mathematics, West Virginia University, Morgantown, WV, USA*

ARTICLE INFO

Article history:
Received 22 September 2009
Accepted 9 August 2010
Available online 2 September 2010

ABSTRACT

A *k-edge-weighting* w of a graph G is an assignment of an integer weight, $w(e) \in \{1, \ldots, k\}$, to each edge e. An edge weighting naturally induces a vertex coloring c by defining $c(u) = \sum_{u\sim e} w(e)$ for every $u \in V(G)$. A k-edge-weighting of a graph G is *vertex-coloring* if the induced coloring c is proper, i.e., $c(u) \neq c(v)$ for any edge $uv \in E(G)$.

Given a graph G and a vertex coloring c_0, does there exist an edge-weighting such that the induced vertex coloring is c_0? We investigate this problem by considering edge-weightings defined on an abelian group.

It was proved that every 3-colorable graph admits a vertex-coloring 3-edge-weighting (Karoński et al. (2004) [12]). Does every 2-colorable graph (i.e., bipartite graphs) admit a vertex-coloring 2-edge-weighting? We obtain several simple sufficient conditions for graphs to be vertex-coloring 2-edge-weighting. In particular, we show that 3-connected bipartite graphs admit vertex-coloring 2-edge-weighting.

1. Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex v of a graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v. If $v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e, $\omega(G)$ denotes the number of connected components of G. An *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \ldots, k$, to the vertices

☆ This work is supported by the Natural Sciences and Engineering Research Council of Canada.
E-mail address: yu@tru.ca (Q. Yu).

1 Tel.: +1 250 3715552; fax: +1 250 3715675.

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2010.08.002

of G, the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two distinct adjacent vertices share the same color. A graph G is *k-colorable* if G has a proper k-vertex coloring. The *chromatic number* $\chi(G)$ is the minimum number r such that G is r-colorable. Notation and terminology that is not defined here may be found in [6].

A *k-edge-weighting* w of a graph G is an assignment of an integer weight $w(e) \in \{1, \dots, k\}$ to each edge e of G. An edge weighting naturally induces a vertex coloring $c(u)$ by defining $c(u) = \sum_{u\sim e} w(e)$ for every $u \in V(G)$. An k-edge-weighting of a graph G is *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and then we say G admitting a *vertex-coloring k-edge-weighting*. There are many variations of vertex-coloring edge-weighting, for instance, a graph G is *vertex-injective* if for any pair of vertices u, v, $c(u) \neq c(v)$. Another concept is irregularity strength, which is a different approach but is similar enough. A multigraph is *irregular* if no two vertex degrees are equal. A multigraph can be viewed as a weighted graph with nonnegative integer weights on the edges. The degree of a vertex in a weighted graph is the sum of the incident weights. Chartrand et al. [9] defined the *irregularity strength* of a graph G to be the minimum of the maximum edge weight in an irregular multigraph with underlying graph G. Other related results and variations can be found in [1,4,5,7] and [10].

If a graph has an edge as a component, clearly it cannot have a vertex-coloring k-edge-weighting. So in this paper, we only consider graphs without a K_2 component and refer to such graphs as *nice graphs*.

In [12], Karoński, Łuczak and Thomason initiated the study of vertex-coloring k-edge-weighting and they brought forward a conjecture as following.

Conjecture 1.1 (*1–2–3-Conjecture*). *Every nice graph admits a vertex-coloring* 3*-edge-weighting.*

Furthermore, they proved that the conjecture holds for 3-colorable graphs (see Theorem 1 in [12]). For other graphs, Addario-Berry et al. [2] showed that every nice graph admits a vertex-coloring 30-edge-weighting. Addario-Berry et al. [3] improved the number of integers required to 16. Later, Wang and Yu [13] improved this bound to 13. Recently, Kalkowski et al. [11] showed that every nice graph admits a vertex-coloring 5-edge-weighting, which is a great leap towards the 1–2–3-Conjecture.

In this paper, we focus on vertex-coloring 2-edge-weighting. In Section 2, we present several new results about vertex-coloring 2-edge-weighting.

Besides the existence problem of vertex-coloring k-edge-weighting, a natural question to ask is that given a graph G and a vertex coloring c_0, can we realize the coloring c_0 by a k-edge-weighting, i.e., does there exist an edge-weighting such that the induced vertex coloring is c_0? For general graphs, it is not easy to find such an edge-weighting. However, by restricting edge weights to an abelian group, we obtain a neat positive answer for this even for a non-proper coloring c_0. In Section 3, we show that every 3-connected nice bipartite graph admits a vertex-coloring 2-edge-weighting.

2. Vertex-coloring 2-edge-weighting

For a graph G, there is a close relationship between 2-edge-weightings and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding a special factor of graphs (see [2,3]). So to find spanning subgraphs with pre-specified degree is an important part of edge-weighting. We shall use some of these results in our proofs.

Lemma 2.1 (*[3]*). *Given a graph $G = (V, E)$, if for all $v \in V$, there are integers a_v^-, a_v^+ such that $a_v^- \leq \left\lfloor \frac{1}{2} d(v) \right\rfloor \leq a_v^+ < d(v)$, and*

$$a_v^+ \leq \min\left\{\frac{1}{2}(d(v) + a_v^-) + 1, 2(a_v^- + 1) + 1\right\},$$

then there exists a spanning subgraph H of G such that $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$.

Given an arbitrary vertex coloring c_0, we want to find an edge-weighting such that the induced vertex coloring is c_0? Under a weak condition, the next two theorems show that there exists an edge-weighting from an abelian group to $E(G)$ to induce c_0 for bipartite and non-bipartite graphs respectively.

Theorem 2.2. *Let G be a non-bipartite graph and $\Gamma = \{g_1, g_2, \ldots, g_k\}$ be a finite abelian group, where $k = |\Gamma|$. Let c_0 be any k-vertex coloring of G with color classes $\{U_1, \ldots, U_k\}$, where $|U_i| = n_i$ $(1 \le i \le k)$. If there exists an element $h \in \Gamma$ such that $n_1g_1 + \cdots + n_kg_k = 2h$, then there is an edge-weighting with the elements of Γ such that the induced vertex coloring is c_0.*

Proof. Let c_0 be any k-vertex coloring with vertex partition $\{U_1, \ldots, U_k\}$, where every element in U_i is colored with g_i $(1 \le i \le k)$ such that $n_1g_1 + \cdots + n_kg_k = 2h$.

Assign one edge with weight h and the rest with zero, so the sum of vertex colors is $2h$. We now adjust this initial weighting, while maintaining the sum of vertex weights, until all the vertices in U_i have color g_i $(1 \le i \le k)$. Suppose there exists a vertex $u \in U_i$ with the wrong color $g \ne g_i$. Since $n_1g_1 + \cdots + n_kg_k = 2h$, there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is non-bipartite, we can choose a walk of even length from u to v, which is always possible since $k \ge 3$. Traverse this walk, adding $g_i - g, g - g_i, g_i - g, \ldots$ alternately to the edges as they are encountered. This operation maintains the sum of vertex weights, leaves the colors of all but u and v unchanged, and yields one more vertex of the correct color. Hence, repeated applications give the desired weighting. $\square$

Note that in Theorem 2.2, the given vertex-coloring c_0 can be either a proper or an improper coloring.

Theorem 2.3. *Let G be a nice bipartite graph and $Z_2 = \{0, 1\}$. Let c_0 be any 2-vertex coloring of G with color classes $\{U_0, U_1\}$, where $|U_i| = n_i$ with $c_0(U_i) = i$ $(i = 0, 1)$. If n_1 is even, then there exists an edge-weighting with the elements of Z_2 such that the induced vertex coloring is c_0.*

Proof. Let $g_1 = 0$ and $g_2 = 1$. If there is a vertex u of color g_i with the wrong color $g \ne g_i$, and since n_2 is even, then there must be another vertex $v \in V(G)$ whose color is also wrong. Since G is connected, then there is a path from u to v. Traverse this walk and add $1, 1, 1, \ldots$ to the edges as they are encountered. This operation always maintains the sum of vertex colors, leaves the colors of all but u and v unchanged, and yields one more vertex of the correct weight. $\square$

Remark. The edge-weighting problem on groups has been studied by Karoński et al. in [12]. They proved that for each $|\Gamma|$-colorable graph G, there exists an edge-weighting with the elements of Γ such that the induced vertex-coloring is proper. Our proofs of Theorems 2.2 and 2.3 are modifications of that result.

For the convenience of applying Theorem 2.3, we restate it in terms of 1, 2 as follows.

Corollary 2.4. *Let G be a nice bipartite graph. Let $U \subseteq V(G)$ and $\overline{U} = V(G) - U$, where $|U| = n_1$ and $\overline{U} = n_2$. If n_1 is even, then there exists an edge-weighting with the elements from $\{1, 2\}$ such that the induced vertex coloring c satisfies that $c(x)$ is odd for all $x \in U$ and $c(y)$ is even for all $y \in \overline{U}$.*

Proof. Let $c_0 : V(G) \to \{0, 1\}$ such that $c_0(U) = \{1\}$ and $c_0(\overline{U}) = \{0\}$. By Theorem 2.3, there exists an edge-weighting, say w, with the elements of Z_2 such that the induced vertex coloring is c_0. Let $w' : E(G) \to \{1, 2\}$ be defined as follows:

$$w'(e) = \begin{cases} 2 & \text{if } w(e) = 0, \\ 1 & \text{if } w(e) = 1. \end{cases}$$

Then w' is a desired edge-weighting. $\square$

It was proved in [12] that every 3-colorable graph has a vertex-coloring 3-edge-weighting. A natural question to ask is whether every 2-colorable graph (i.e., bipartite graphs) has a vertex-coloring 2-edge-weighting. But the answer is no, since C_6 and C_{10} do not admit vertex-coloring 2-edge-weightings. In fact, Chang et al. [8] proved the following results.

Lemma 2.5 (*[8]*). *Every connected nice bipartite graph admits a vertex-coloring 2-edge-weighting if one of following conditions holds:*

(1) $|A|$ *or* $|B|$ *is even;*
(2) $\delta(G) = 1$*;*
(3) $\lfloor d(u)/2 \rfloor + 1 \ne d(v)$ *for any edge* $uv \in E(G)$.

An interesting corollary of Lemma 2.5 is that every r-*regular* nice bipartite graph ($r \geq 3$) admits a vertex-coloring 2-edge-weighting. More generally, every bipartite $[r, r+1]$-graph G (i.e., $d_G(v) \in \{r, r+1\}$ for any $v \in V(G)$) with $r \geq 4$ admits a vertex-coloring 2-edge-weighting.

Theorem 2.6. *Let G be a nice graph. If $\delta(G) \geq 8\chi(G)$, then G admits a vertex-coloring 2-edge-weighting.*

Proof. Let $\{V_1, \ldots, V_{\chi(G)}\}$ be a partition of $V(G)$ into independent sets. For each $v \in V_i$, choose a_v^- such that $\left\lfloor \frac{d(v)}{4} \right\rfloor \leq a_v^- \leq \left\lfloor \frac{d(v)}{2} \right\rfloor$, $a_v^- + d_G(v) \equiv 2\mathrm{i} \pmod{2\chi(G)}$, and $a_v^- + 2\chi(G) \geq \left\lfloor \frac{d(v)}{2} \right\rfloor$. Such choice for a_v^- exists as $\delta(G) \geq 8\chi(G)$. Set $a_v^+ = a_v^- + 2\chi(G)$.

Furthermore, such a choices of a_v^- and a_v^+ satisfy the conditions of Lemma 2.1, i.e.,

$$\begin{aligned}\frac{1}{2}(d(v) - a_v^- - 2\chi(G)) - \chi(G) &= \frac{1}{2}(d(v) - a_v^+) - \chi(G) \\ &\geq \frac{d(v)}{8} - \chi(G),\end{aligned}$$

thus there is a subgraph H such that for all v, $d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}$. Set $w(e) = 2$ for $e \in E(H)$ and $w(e) = 1$ for $e \in E(G) - E(H)$. If $v \in V_i$, we have

$$\sum_{v \sim e} w(e) = 2d_H(v) + d_{G-H}(v) = d_G(v) + d_H(v) \in \{2\mathrm{i}, 2\mathrm{i}+1\} \pmod{2\chi(G)}.$$

Thus adjacent vertices in different parts of $\{V_1, \ldots, V_{\chi(G)}\}$ have different arities. As each V_i is an independent set, these weights form a vertex-coloring 2-edge-weighting of G. □

Theorem 2.7. *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v - N(v)$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.5, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$. Since $G - v - N(v)$ is connected and $|U - v|$ is even, by Corollary 2.4, $G - v - N(v)$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W - N(v)$. Now we assign every edge of $E[N(v), U]$ with weight 2. Clearly $c(x)$ is odd for all $x \in U - v$ and $c(y)$ is even for all $y \in W$. Note that $c(v) = 2d_G(v)$ and $c(u) = 2d_G(u)$ for all $u \in N(v)$. Since $d_G(u) \neq d_G(v)$, so $c(v) \neq c(u)$ for all $u \in N(v)$. Thus we obtain a vertex-coloring 2-edge-weighting of G. □

Theorem 2.8. *Given a nice bipartite graph $G = (U, W)$, if there exists a vertex v of degree $\delta(G)$ such that $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$ and $G - v$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If $|U| \cdot |W|$ is even, by Lemma 2.5, the result follows. So we may assume that both $|U|$ and $|W|$ are odd. Let $v \in U$ satisfy $d_G(v) = \delta(G)$ and $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$. Now we consider two cases.

Case 1. $\delta(G)$ is even.

In this case, $|(U - v) \cup N(v)|$ is even and $|W - N(v)|$ is odd. By Corollary 2.4, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in (U - v) \cup N(v)$ and $c(y)$ is even for all $y \in W - N(v)$. Assigning the edges incident to v with weight 1. Then $c(w)$ is even for all $w \in W$ and $c(u)$ is odd for all $u \in U - v$. Note that $c(v) = \delta(G) < d_G(u) \leq c(u)$ for all $u \in N(v)$, so we obtain a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

In this case, $|(U - v) \cup N(v)|$ is odd and $|W - N(v)|$ is even. By Corollary 2.4, $G - v$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in (U - v) \cup N(v)$ and $c(y)$ is odd for all $y \in W - N(v)$. Since $d_G(v) \notin \{d_G(x) \mid x \in N(v)\}$, similar to Case 1, assigning the edges incident to v with weight 1 induces a vertex-coloring 2-edge-weighting of G. □

3. 3-connected bipartite graphs

In this section, we continue the research in this direction and prove that there exists a vertex-coloring 2-edge-weighting in every 3-connected bipartite graph. The following lemma is an important step in proving this result.

Lemma 3.1. *Let G be a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $t \leq \delta - 1$. Denote $N^{\delta}(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$. Then there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ ($i = 1, \ldots, t$), such that $G - u - \{e_1, \ldots, e_t\}$ is connected.*

Proof. Let $C_1, \ldots, C_s$ be the components of $G - u - N^{\delta}(u)$. If we shrink each component C_i to a vertex c_i, then we obtain a bipartite multi-graph $H = (X, Y)$ associated with $G - u$ as follows:
$X = \{u_1 \ldots, u_t\}$, $Y = \{c_1, \ldots, c_s\}$ and $|E_H(u_i, c_j)| = |E_G(u_i, C_j)|$ for $1 \leq i \leq t$ and $1 \leq j \leq s$.
Clearly, $d_H(u_i) = \delta - 1$ for every $u_i \in X$.

Claim. *H contains a connected spanning subgraph T such that $d_T(v) \leq \delta - 2$ for every $v \in X$.*

Suppose that the claim does not hold. Let R be a connected induced subgraph of H satisfying

(i) R contains a connected spanning subgraph M such that $d_M(v) \leq \delta - 2$ for every $v \in V(M) \cap X$;
(ii) $|V(R)|$ is maximum.

It is easy to see that $V(R) \neq \emptyset$ and $R \neq H$. Let $R = (A, B)$, where $A \subseteq X$ and $B \subseteq Y$. By the maximality of R, we have $d_R(v) \geq \delta - 2$ for every $v \in A$ and $E_H(B, X - A) = \emptyset$. Let $L = \{v \mid d_R(v) = \delta - 2, v \in A\}$. We see $|L| \geq 2$ since G is 3-connected. Let M^* be a connected spanning subgraph of R such that $d_{M^*}(v) = \delta - 2$ for every $v \in A$. Note that for every connected spanning subgraph N^* of M^*, we have $d_{N^*}(w) = \delta - 2$ for $w \in L$ by the maximality of R. So every edge incident with w in M^*, where $w \in L$, is a cut-edge of M^*. Let $|L| = l$ and $|E(R) - E(M^*)| = m$. Then $l + m \leq t \leq \delta - 1$. We have

$$\omega(M^* - L) = \omega(H - L - (E(R) - E(M^*))) - 1 \geq (\delta - 3)l + 1.$$

So $\omega(H - L) \geq (\delta - 3)l + 2 - m$, which implies

$$\begin{aligned} \omega(G - u - L) &\geq (\delta - 3)l + 1 - m + 1 \\ &\geq (\delta - 3)l + 2 - (\delta - 1 - l) \\ &= (\delta - 2)l + 3 - \delta. \end{aligned}$$

Since G is 3-connected, then

$$3\omega(G - u - L) \leq (\delta - 1)l + \delta - l.$$

It follows that

$$\omega(G - u - L) \leq \left\lfloor \frac{(\delta - 1)l + \delta - l}{3} \right\rfloor.$$

However

$$(\delta - 2)l + 3 - \delta - \frac{(\delta - 1)l + \delta - l}{3} = \frac{2\delta l}{3} - \frac{4\delta}{3} - \frac{4l}{3} + 3 > 0,$$

a contradiction. So we complete the claim and thus obtain a connected spanning subgraph T of H.

Let E' denote the set of corresponding edges of $E(T)$ in G. Then we obtain a spanning subgraph $T^* = \bigcup_{i=1}^{s} C_i \cup N^{\delta}(u) \cup E'$ of $G - u$ such that $d_{T^*}(v) \leq \delta - 2$ for every $v \in N^{\delta}(u)$. Thus the proof is complete. □

The following theorem is the main result of this section.

Theorem 3.2. *Let $G = (U, W)$ be a nice bipartite graph. If G is 3-connected, then G admits a vertex-coloring 2-edge-weighting.*

Proof. If G is a regular graph, the result follows from Lemma 2.5(3). So we may assume that G is a 3-connected non-regular bipartite graph with bipartition (U, W). Let $u \in U$ with $d(u) = \delta(G)$ and $N^{\delta}(u) = \{v \mid d(v) = \delta, v \in N_G(u)\} = \{u_1, \ldots, u_t\}$, where $t \le \delta - 1$. Then by Lemma 3.1, there exist $e_1, \ldots, e_t$, where e_i is incident to vertex u_i in $G - u$ for $i = 1, \ldots, t$, such that $G - u - \{e_1, \ldots, e_t\}$ is connected.

By Lemma 2.5, we can assume that $|U|\,|W|$ is odd. Now we consider two cases.

Case 1. $\delta(G)$ is even.

Then $|N(u) \cup (U - u)|$ is even. By Corollary 2.4, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is odd for all $x \in N(u) \cup (U - u)$ and $c(y)$ is even for all $y \in W - N(u)$. We assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. Then $c(x)$ is odd for all $x \in U - u$, and $c(y)$ is even for all $y \in W$. Moreover, $c(u_i) > d(u_i) = d(u) = c(u)$ for all $i = 1, \ldots, t$. Note that $c(y) \ge d(y) > d(u) = c(u)$ for all $y \in N(u) - N^{\delta}(u)$. Hence we obtain a vertex-coloring 2-edge-weighting of G.

Case 2. $\delta(G)$ is odd.

Then $|W - N(u)|$ is even. By Corollary 2.4, $G - u - \{e_1, \ldots, e_t\}$ has a vertex-coloring 2-edge-weighting such that $c(x)$ is even for all $x \in N(u) \cup (U - u)$ and $c(y)$ is odd for all $y \in W - N(u)$. We again assign every edge of $\{e_1, \ldots, e_t\}$ with weight 2 and every edge of $\{uu_i \mid i = 1, \ldots, t\}$ with weight 1. Similar to Case 1, $c(u) = d(u)$ and $c(u) < c(u_i)$ for $i = 1, \ldots, t$. Moreover, $c(u_i)$ is odd for $i = 1, \ldots, t$. Then we obtain a vertex-coloring 2-edge-weighting of G.

We complete the proof. □

Based on the proof of Theorem 3.2, we can easily obtain the following corollary.

Corollary 3.3. *Let $G = (U, W)$ be a bipartite graph with $\delta(G) \ge 3$. If there exists a vertex of degree $\delta(G)$ such that $G - u - N(u)$ is connected, then G admits a vertex-coloring 2-edge-weighting.*

4. Conclusions

Karoński et al. [12] showed that for any fixed $p \in (0, 1)$ the random graph $G_{n,p}$ of order *n almost surely* admits a vertex-coloring 2-edge-weighting. That is, the edges of almost all graphs can be labeled with 1 or 2 to induce a proper vertex-coloring. So a natural question is to classify all graphs which admit vertex-coloring 2-edge-weighting.

As an initial step towards this investigation, one may study bipartite graphs first. Since there exist families of infinite bipartite graphs (e.g., the generalized θ-graphs) which only admit vertex-coloring 3-edge-weightings, it is nontrivial to classify all bipartite graphs with vertex-coloring 2-edge-weighting. In light of Theorem 3.2, it remains an open problem to classify all 2-connected bipartite graphs which admit a vertex-coloring 2-edge-weighting.

Acknowledgements

The authors are indebted to Dr. Yinghua Duan for the valuable discussion. The authors also express gratitude to Nankai University, where this research was carried out, for the support and hospitality.

References

[1] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed, Vertex coloring edge partitions, J. Combin. Theory Ser. B 94 (2005) 237–244.
[2] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, A. Thomason, Vertex-coloring edge-weightings, Combintorica 27 (2007) 1–12.
[3] L. Addario-Berry, K. Dalal, B.A. Reed, Degree constrained subgraphs, Discrete Appl. Math. 156 (2008) 1168–1174.
[4] M. Aigner, E. Triesch, Zs. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs, in: A. Barlotti, et al. (Eds.), Combinatorics, vol. 90, Elsevier Science Pub., New York, 1992, pp. 1–9.
[5] P.N. Balister, O.M. Riordan, R.H. Schelp, Vertex-distinguishing edge colorings of graphs, J. Graph Theory 42 (2003) 95–109.
[6] B. Bollobás, Modern Graph Theory, 2nd ed., Springer-Verlag New York, Inc., 1998.
[7] A.C. Burris, R.H. Schelp, Vertex-distinguishing proper edge colorings, J. Graph Theory 26 (1997) 73–82.
[8] G.J. Chang, C. Lu, J. Wu, Q.L. Yu, Vertex coloring 2-edge weighting of bipartite graphs, Taiwanese J. Math. (in press).

[9] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. Numer. 64 (1988) 197–210.
[10] K. Edwards, The harmonioous chromatic number of bounded degree graphs, J. Lond. Math. Soc. 55 (1997) 435–447.
[11] M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: towards the 1–2–3-conjecture, J. Combin. Theory Ser. B 100 (2010) 347–349.
[12] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colors, J. Combin. Theory Ser. B 91 (2004) 151–157.
[13] T. Wang, Q.L. Yu, On vertex-coloring 13-edge-weighting, Front. Math. China 3 (2008) 581–587.