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Chords of longest circuits in locally planar graphs

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Abstract

It was conjectured by Thomassen ([B. Alspach, C. Godsil, Cycle in graphs, Ann. Discrete Math. 27 (1985)], p. 466) that every longest circuit of a 3-connected graph must have a chord. This conjecture is verified for locally 4-connected planar graphs, that is, let N be the set of natural numbers; then there is a function $h : N \to N$ such that, for every 4-connected graph G embedded in a surface S with Euler genus g and face-width at least $h(g)$, every longest circuit of G has a chord.

1. Introduction

An edge e is called a *chord* of a circuit if e is not an edge of the circuit and both end-vertices of e are in the circuit.

The following conjecture was proposed by Thomassen.

Conjecture 1.1 (*Thomassen ([1], p. 466)*). *Every longest circuit of a 3-connected graph must have a chord.*

Conjecture 1.1 has been verified for the following families of graphs.

1. (Zhang [16]) 3-connected cubic planar graphs and 3-connected planar graphs with minimum degree at least four.

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2. (Thomassen [13]) 3-connected cubic graphs.
3. (Li and Zhang [6]) 3-connected projective planar graphs with minimum degree at least four.
4. (Li and Zhang [7]) 4-connected graphs embedded in the torus or the Klein bottle.

In this paper, we will verify the conjecture for 4-connected *locally planar graphs*, i.e., 4-connected graphs on a fixed surface with sufficiently large face-width.

Theorem 1.2. *Let N be the set of natural numbers. There is a function $h : N \to N$, such that, for every 4-connected graph G embedded in a surface S with Euler genus g and face-width at least $h(g)$, every longest circuit of the graph G has a chord.*

It is well known that every 4-connected planar graph is hamiltonian (by Tutte [14]). So clearly every longest circuit in a 4-connected planar graph has a chord. But as far as we know, it is not known that every longest circuit in a 3-connected planar graph or a 3-connected graph on a fixed surface has a chord. Our result may be considered as a 4-connected version of this question. It is known that every 4-connected graph in the projective plane is hamiltonian [10], so Theorem 1.2 gives a weaker result for projective planar graphs. Also, there is a Grünbaum–Nash-Williams conjecture [4,9] which says that every 4-connected graph in the torus is hamiltonian. So Theorem 1.2 would be a weaker result for toroidal graphs although the Grünbaum–Nash-Williams conjecture [4,9] is still open.

However, it was observed by Thomassen [12] that no matter how large the face-width is, there are 4-connected triangulations which are not 1-tough. Hence, Theorem 1.2 does make sense for graphs on a higher surface.

It was conjectured by Yu [15] (also in [8]), that every 5-connected graph on a fixed surface with sufficiently large face-width is hamiltonian. This conjecture was verified for toroidal graphs [11] and triangulations [15]. Actually, Thomas and Yu [11] proved that every 5-connected toroidal graph is hamiltonian, and Kawarabayashi [5] extended Yu's result and proved that every 5-connected triangulation on a fixed surface with sufficiently large face-width is hamiltonian-connected.

2. Terminology and notation

All graphs considered in this paper are simple graphs. That is, they are finite, undirected, and without loops or multiple edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively.

Let G be a graph, and let $u, v \in V(G)$. The vertex u is a *neighbor* of v if $uv \in E(G)$. The set of all neighbors of v is denoted by $N(v)$. The degree of a vertex v, denoted by $d_G(v)$, is the number of neighbors of v.

All embeddings of graphs in surfaces considered in this paper are 2-cell embedding unless there is a specification. If G is embedded in a surface S with f faces, then the number $g = 2 - |V(G)| + |E(G)| - f$ is called the *Euler genus* of S. The *edge-width* **ew(G)** of a graph G embedded in a non-simply connected surface is defined as the length of a shortest non-contractible circuit in G. The *face-width* or *representativity*, denoted by **fw(G)**, is the minimum k such that every non-contractible simple closed curve on the surface intersects G in at least k points.

Most standardized notation can be found in books, such as [3,8].

3. Lemmas

The following is the key lemma for the proof of the main theorem of the paper.

Lemma 3.1 (*Böhme et al. [2]*). *Let N be the set of natural numbers, let R^+ be the set of positive real numbers. There is a function $f : N \times R^+ \to N$ such that for every $\epsilon > 0$ and every 4-connected graph G of order n embedded in a surface with Euler genus g and face-width at least $f(g, \epsilon)$, G contains a circuit C with $|V(C)| > (1 - \epsilon)n$.*

Lemma 3.2. *Let G be a connected graph with $|V(G)| \geq 3$ embedded in a surface S with Euler genus g, then $|E(G)| < 3|V(G)| + 3g$.*

Proof. Assume G is embedded in S with f faces. Since each face is bounded by at least 3 edges, $3f \leq 2|E(G)|$. By the Euler formula,

$$2 - g = |V(G)| - |E(G)| + f \leq |V(G)| - |E(G)| + \frac{2|E(G)|}{3}.$$

That is,

$$2 - g \leq |V(G)| - \frac{|E(G)|}{3}.$$

Hence,

$$|E(G)| \leq 3|V(G)| - 6 + 3g < 3|V(G)| + 3g. \quad \square$$

4. Proof of the main theorem

The main theorem (Theorem 1.2) is proved in this section.

Choose $\epsilon \leq \frac{1}{64}$. By Lemma 3.1, there is a function $f : N \times R^+ \to N$ such that, for every 4-connected graph G embedded in a surface with Euler genus g and face-width at least $f(g, \epsilon)$, every longest circuit of G is of length at least

$$(1 - \epsilon)|V(G)| \geq \frac{63}{64}|V(G)|. \tag{1}$$

Let $h(g) = \max\{f(g, \epsilon), 30g\}$. And let G be a 4-connected graph embedded in a surface S with Euler genus g and face-width at least $h(g)$. We will verify Theorem 1.2 for the graph G by way of contradiction.

Let C be a longest circuit of G without a chord, let G' be a graph embedded in S obtained from G by deleting all edges of C and all edges joining two vertices in $V(G) - V(C)$. Let $A = V(G) - V(C)$. It is obvious that G' is bipartite with the bipartition $\{V(C), A\}$.

Let f' be the number of faces of G' embedded in S, and f'_4 be the number of faces of G' in S bounded by 4 edges. Note that the embedding of G' in surface S is not necessarily a 2-cell embedding.

Recall that the embedding of G' in S is inherited from the embedding of G in S by deleting edges. As we know, after deleting one edge, the number of faces remains the same or is deceased by one. Hence, we have

$$2 - g \leq |V(G')| - |E(G')| + f'. \tag{2}$$

I. We claim that

$$|E(G')| \leq \frac{11}{5}|V(C)|. \tag{3}$$

For otherwise, we assume that $|E(G')| > \frac{11}{5}|V(C)|$. Since C is chordless, the following equality holds:

$$|E(G')| = \sum_{v \in V(C)} (d_G(v) - 2). \tag{4}$$

Furthermore, by the construction of the bipartite graph G',

$$|E(G')| \leq |E(G) - E(C)| \leq \sum_{v \in A} d_G(v). \tag{5}$$

Combine inequalities (4) and (5);

$$\sum_{v \in V(C)} (d_G(v) - 2) \leq \sum_{v \in A} d_G(v). \tag{6}$$

The inequality (6) implies that

$$\sum_{v \in V(C)} (2d_G(v) - 2) \leq \sum_{v \in V(G)} d_G(v). \tag{7}$$

Note that $\sum_{v \in V(G)} d(v) = 2|E(G)|$. Hence, by (7),

$$\sum_{v \in V(C)} (d_G(v) - 1) \leq |E(G)|$$

and therefore,

$$|V(C)| + \sum_{v \in V(C)} (d_G(v) - 2) \leq |E(G)|. \tag{8}$$

Substituting (4) into (8), we have that

$$|V(C)| + |E(G')| \leq |E(G)|. \tag{9}$$

Since $|E(G')| > \frac{11}{5}|V(C)|$, we have that

$$\frac{16}{5}|V(C)| \leq |E(G)|. \tag{10}$$

Therefore, by Lemma 3.2 and (10),

$$\frac{16}{5}|V(C)| \leq |E(G)| < 3|V(G)| + 3g. \tag{11}$$

Since the face-width of G is at least $h(g) \geq 30g$, we have that $|V(G)| \geq 30g$. Therefore, by (11),

$$\frac{16}{5}|V(C)| < 3|V(G)| + \frac{|V(G)|}{10} = \frac{31|V(G)|}{10}.$$

That is,

$$|V(C)| < \frac{31|V(G)|}{32}.$$

Note that C is a longest circuit of G. By Lemma 3.1 (or (1)),

$$|V(C)| > (1-\epsilon)|V(G)| \geq \frac{63|V(G)|}{64}.$$

This is a contradiction. $\square$

II. We claim that

$$f_4' \geq 4|V(C)| - 3|V(G)| - 3g. \tag{12}$$

Since G' is bipartite and simple, each face is bounded by at least 4 edges. Hence,

$$4f_4' + 6(f' - f_4') \leq 2|E(G')|. \tag{13}$$

Simplifying (13),

$$3f' - f_4' \leq |E(G')|,$$

and therefore,

$$f' \leq \frac{1}{3}|E(G')| + \frac{1}{3}f_4'. \tag{14}$$

On the other hand, note that (by (2))

$$2 - g \leq |V(G')| - |E(G')| + f'. \tag{15}$$

Substituting (14) into (15), we have that

$$2 - g \leq |V(G')| - |E(G')| + f' \leq |V(G')| - |E(G')| + \frac{1}{3}|E(G')| + \frac{1}{3}f_4'.$$

That is,

$$2 - g \leq |V(G')| - \frac{2}{3}|E(G')| + \frac{1}{3}f_4'. \tag{16}$$

Rearranging the inequality (16), we have that

$$f_4' \geq 2|E(G')| - 3|V(G')| + 6 - 3g > 2|E(G')| - 3|V(G')| - 3g. \tag{17}$$

Moreover, since G is 4-connected, $d_G(v) \geq 4$ for every $v \in V(G)$. Therefore, $d_{G'}(v) \geq 2$ for $v \in V(C)$. Hence, by the construction of the bipartite graph G',

$$|E(G')| \geq 2|V(C)|. \tag{18}$$

Combining (17) and (18), we have that

$$f_4' \geq 2|E(G')| - 3|V(G')| - 3g \geq 4|V(C)| - 3|V(G)| - 3g. \quad \square$$

III. We claim that, *in the graph G', every vertex $v \in V(C)$ is incident with at most $(d_{G'}(v) - 1)$ 4-faces.*

Let $C = v_1 \cdots v_r v_1$. Assume that some vertex $v_i \in V(C)$ is incident with $d_{G'}(v)$ 4-faces. We claim that we may assume that the edge $v_i v_{i+1} (\in E(C) \subseteq E(G) - E(G'))$ lies inside a 4-face $R = v_i x v_{i+1} y v_i$ of G' where $x, y \notin V(C)$.

To see the claim, let $u_1, \ldots, u_d$ be the neighbor of v_i in G' with $d = d_{G'}(v_i)$. Let $u_{d+1} = u_1$. Then there are d vertices $u_1', \ldots, u_d'$ in $V(C) - \{v_i\}$ such that u_i' is adjacent to both u_i and u_{i+1} for $i = 1, \ldots, d$ in G'. Since the edge-width of G' is still large (otherwise, the edge-width of the original graph G is small, a contradiction; note that $ew(G) \geq fw(G)$), the graph induced by $\{v_i, u_1, \ldots, u_d, u'_1, \ldots, u'_d\}$ in G is planar. Since v_i is surrounded by 4-cycles in G', this implies $v_{i-1} \in \{u_1, \ldots, u_d\}$.

Note that the circuit C could be extended around the 4-circuit R by inserting x (or y) between v_i and v_{i+1} in G. This contradicts C being a longest circuit of G. □

IV. We claim that

$$f'_4 \leq \frac{3}{5}|V(C)|. \tag{19}$$

Since G' is bipartite, every face bounded with 4 edges must contain exactly two vertices of $V(C)$. By Claim III, we have that

$$f'_4 \leq \sum_{v \in V(C)} \frac{1}{2}(d_{G'}(v) - 1).$$

That is,

$$f'_4 \leq \frac{1}{2}|E(G')| - \frac{1}{2}|V(C)|. \tag{20}$$

By Claim I, we have $|E(G')| \leq \frac{11}{5}|V(C)|$. Hence, the inequality (20) implies that

$$f'_4 \leq \frac{1}{2}|E(G')| - \frac{1}{2}|V(C)| \leq \frac{3}{5}|V(C)|. \quad \square$$

V. The final step.

By Claims II and IV, we have (by combining (12) and (19))

$$4|V(C)| - 3|V(G)| - 3g \leq \frac{3}{5}|V(C)|. \tag{21}$$

Simplifying (21),

$$\frac{17}{5}|V(C)| \leq 3|V(G)| + 3g. \tag{22}$$

Note that $|V(G)| \geq 30g$ since the face-width of G is at least $h(g) \geq 30g$. Therefore, the inequality (22) implies that

$$\frac{17}{5}|V(C)| \leq 3|V(G)| + 3g \leq \frac{31}{10}|V(G)|. \tag{23}$$

Simplifying (23),

$$|V(C)| \leq \frac{31}{34}|V(G)| < \frac{31}{32}|V(G)|. \tag{24}$$

Note that C is a longest circuit of G. By Lemma 3.1 (or (1)), $|V(C)| > (1-\epsilon)|V(G)| \geq \frac{63|V(G)|}{64}$. This is a contradiction and completes the proof of the main theorem. □

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