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# Flow-contractible configurations and group connectivity of signed graphs

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ABSTRACT

Jaeger, Linial, Payan and Tarsi (JCTB, 1992) introduced the concept of group connectivity as a generalization of nowhere-zero flow for graphs. In this paper, we introduce group connectivity for signed graphs and establish some fundamental properties. For a finite abelian group $A$, it is proved that an $A$-connected signed graph is a contractible configuration for $A$-flow problem of signed graphs. In addition, we give sufficient edge connectivity conditions for signed graphs to be $A$-connected and study the group connectivity of some families of signed graphs.



## 1. Introduction

The notion of nowhere-zero flows of ordinary graphs was introduced by Tutte [15,16] as a dual problem to vertex coloring of graphs embedded on an orientable surface. The definition of nowhere-zero flows of signed graphs naturally comes from the study of embeddings of graphs in non-orientable surfaces, where nowhere-zero flows emerge as the dual notion to local tensions.

The group connectivity, as a generalization of the flow problem, is a concept introduced by Jaeger, Linial, Payan and Tarsi [5]. Furthermore, graphs with certain group connectivity are contractible configurations for flow problems.

In this paper, the concept and results about group connectivity [5] for ordinary graphs are extended to signed graphs.

### 1.1. Group connectivity for ordinary graphs

Throughout the paper, we consider finite graphs. Loops and multiple edges are allowed. We refer [21] for undefined notations and terminology on nowhere-zero flows.

Let $A$ be a non-trivial (additive) abelian group with additive identity 0, and let $A^* = A\backslash\{0\}$ be the set of nonzero elements in $A$. Let $D$ be an orientation of $G$. Define $F(G, A) = \{f \,|\, f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f \,|\, f : E(G) \mapsto A^*\}$. For each $f \in F(G, A)$, the *boundary* of $f$ is the function $\partial f : V(G) \mapsto A$ defined by $\partial f(v) = \sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e)$ for each vertex $v \in V(G)$. $(D, f)$ is called an *$A$-flow* if $\partial f = 0$, and is called a *nowhere-zero $A$-flow* if moreover $f \in F^*(G, A)$. If $A = \mathbb{Z}$ and $1 \le |f(e)| \le k-1$ for each $e \in E(G)$, the flow $(D, f)$ is called a *nowhere-zero $k$-flow*. Tutte's flow conjectures are some of the major open problems in graph theory. The 3-flow conjecture states that *every 4-edge-connected graph admits a nowhere-zero 3-flow* and the 5-flow

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conjecture claims that *every bridgeless graph admits a nowhere-zero* 5*-flow*. The readers are referred to [9] for a recent survey on this topic.

Jaeger, Linial, Payan and Tarsi [5] introduced the concept of *group connectivity* as a generalization of nowhere-zero flows of graphs. It is obvious that $\sum_{v\in V(G)}\partial f(v) = 0$ for any $f \in F^*(G, A)$. This motivates the definition of *A-boundary function*. A mapping $b : V(G) \mapsto A$ is called an *A-boundary* of $G$ if $\sum_{v\in V(G)} b(v) = 0$. Let $Z(G, A)$ be the collection of all $A$-boundaries of $G$. $G$ is *A-connected* if, for any $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$, that is, for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{e\in E^+_D(v)} f(e) - \sum_{e\in E^-_D(v)} f(e) = b(v).$$

Jaeger et al. [5] conjectured that *every* 5*-edge-connected graph is* $\mathbb{Z}_3$*-connected,* and *every* 3*-edge-connected graph is* $\mathbb{Z}_5$*-connected*. These two conjectures imply Tutte's 3-flow conjecture and 5-flow conjecture, respectively. Jaeger et al. [5] proved that every 4-edge-connected graph is $A$-connected for any abelian group $A$ with $|A| \geq 4$. Thomassen's breakthrough result in [14] confirmed the conjecture of Jaeger et al. for 8-edge-connected graphs, and it was later improved by Lovász et al. [10] that *every* 6*-edge-connected graph is* $\mathbb{Z}_3$*-connected.* In this paper, we will introduce the concept of group connectivity for signed graphs and extend the above mentioned results to signed graphs with slightly higher edge-connectivity.

### 1.2. Preliminary for signed graphs

A *signed graph* is a graph $G$ with a mapping $\sigma : E(G) \mapsto \{1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. The mapping $\sigma$, called signature, is sometimes implicit in the notation of a signed graph and will be specified when needed. Both negative and positive loops are allowed in signed graphs, while positive loops do not affect any flow property. We use $E^+_\sigma(G)$ and $E^-_\sigma(G)$ to denote the set of positive edges and the set of negative edges in $G$, respectively. If no confusion occurs, we simply use $E^+_\sigma$ for $E^+_\sigma(G)$ and $E^-_\sigma$ for $E^-_\sigma(G)$. An *orientation* $\tau$ assigns each edge of $(G, \sigma)$ as follows: if $e = xy$ is a positive edge, then the edge is either oriented away from $x$ and toward $y$ or away from $y$ and toward $x$; if $e = xy$ is a negative edge, then the edge is oriented either away from both $x$ and $y$ or towards both $x$ and $y$. We call $e = xy$ a *sink edge* (a *source edge*, respectively) if it is oriented away from (towards, respectively) both $x$ and $y$.

Let $\tau$ be an orientation of $(G, \sigma)$. For each vertex $v \in V(G)$, let $H_G(v)$ be the set of half edges incident with $v$. Define $\tau(h) = 1$ if the half edge $h \in H_G(v)$ is oriented away from $v$, and $\tau(h) = -1$ if the half edge $h \in H_G(v)$ is oriented towards $v$. Denote $d^+_\tau(v) = |E^+_\tau(v)|$ ($d^-_\tau(v) = |E^-_\tau(v)|$, respectively) to be the outdegree (indegree, respectively) of $(G, \sigma)$ under orientation $\tau$, where $E^+_\tau(v)$($E^-_\tau(v)$, respectively) denotes the set of outgoing (ingoing, respectively) half edges incident with $v$.

The *switch operation* $\zeta$ on an edge-cut $S$ is a mapping $\zeta : E(G) \mapsto \{-1, 1\}$ such that $\zeta(e) = -1$ if $e \in S$ and $\zeta(e) = 1$ otherwise. Two signatures $\sigma$ and $\sigma'$ are *equivalent* if there exists an edge-cut $S$ such that $\sigma(e) = \sigma'(e)\zeta(e)$ for every edge $e \in E(G)$, where $\zeta$ is the switch operation on the edge-cut $S$. For a signed graph $(G, \sigma)$, let $\mathcal{X}$ denote the collection of all signatures equivalent to $\sigma$. The *negativeness* of $(G, \sigma)$ is denoted by $\epsilon_N(G, \sigma) = \min\{|E^-_{\sigma'}(G)| : \forall\sigma' \in \mathcal{X}\}$. We use $\epsilon_N$ for short if the signed graph $(G, \sigma)$ is understood from the context. A signed graph is called *k-unbalanced* if $\epsilon_N \geq k$. Note that 1-unbalanced signed graph is also known as unbalanced signed graph.

A circuit is *balanced* if $\epsilon_N = 0$ and is *unbalanced* otherwise (i.e. $\epsilon_N = 1$). A signed graph $(G, \sigma)$ is called a barbell if either

- $G$ consists of two unbalanced circuits $C_1, C_2$ with $|V(C_1) \cap V(C_2)| = 1$, or
- $G$ consists of two vertex disjoint unbalanced circuits $C_1, C_2$ and a path $P$, which has one end in $V(C_1)$ and one end in $V(C_2)$ and has no interior vertices in $V(C_1) \cup V(C_2)$.

A *signed circuit* is either a balanced circuit or a barbell.

The signature is usually implicit in the notation of a signed graph if no confusion occurs. We define *contraction* in signed graphs as follows. For an edge $e \in E(G)$, the *contraction* $G/e$ is the signed graph obtained from $G$ by identifying the two ends of $e$, and then deleting the resulting positive loop if $e \in E^+_\sigma$, but keeping the resulting negative loop if $e \in E^-_\sigma$, For $X \subseteq E(G)$, the *contraction* $G/X$ is the signed graph obtained from $G$ by contracting all edges in $X$. If $H$ is a subgraph of $G$, we use $G/H$ for $G/E(H)$. An immediate observation is that the contraction operation does not decrease negativeness. That is, $\epsilon_N(G/H) \geq \epsilon_N(G)$ for any subgraph $H$ of $G$.

### 1.3. Group connectivity of signed graphs

Let $A$ be an abelian group, $2A = \{2\alpha : \alpha \in A\}$, and $A^* = A \setminus \{0\}$. For a signed graph $G$, we still denote $F(G, A) = \{f | f : E(G) \mapsto A\}$ and $F^*(G, A) = \{f | f : E(G) \mapsto A^*\}$. Let $\tau$ be an orientation of $(G, \sigma)$. For each $f \in F(G, A)$, the *boundary* of $f$ is the function $\partial f : V(G) \mapsto A$ defined by

$$\partial f(v) = \sum_{h\in H_G(v)} \tau(h)f(e_h),$$

where $e_h$ is the edge of $G$ containing $h$ and "$\sum$" refers to the addition in $A$. If $\partial f = 0$, then $(\tau, f)$ is called an *A-flow* of $G$. In addition, $(\tau, f)$ is a *nowhere-zero A-flow* if $f \in F^*(G, A)$ and $\partial f = 0$.

For any $f \in F(G, A)$, each positive edge contributes 0, each sink edge $e$ contributes $2f(e)$, and each source edge $e$ contributes $-2f(e)$ to $\sum_{v \in V(G)} \partial f(v)$. Thus we have

$$\sum_{v \in V(G)} \partial f(v) = \sum_{e \text{ is a sink edge}} 2f(e) - \sum_{e \text{ is a source edge}} 2f(e) \in 2A. \tag{1}$$

In particular, if $G$ is an ordinary graph, that is $E_\sigma^- = \emptyset$, then $\sum_{v \in V(G)} \partial f(v) = 0$ for any $f \in F(G, A)$. This motivates the zero-sum $A$-boundary function in the group connectivity of ordinary graphs defined by Jaeger et al. [5] as introduced earlier.

For signed graph with $E_\sigma^- \neq \emptyset$, $\sum_{v \in V(G)} \partial f(v)$ may not be zero but is always equal to $2\alpha$ for some element $\alpha \in A$ by Eq. (1). We introduce the following definition of *group connectivity of signed graphs*.

**Definition 1.1** (*Group Connectivity of Signed Graphs*). Let $(G, \sigma)$ be a 2-unbalanced signed graph with orientation $\tau$ and $A$ be an abelian group.

(i) A mapping $b : V(G) \mapsto A$ is called an *A-boundary* of $(G, \sigma)$ if

$$\sum_{v \in V(G)} b(v) = 2\alpha \text{ for some } \alpha \in A.$$

Let $Z(G, A)$ be the collection of all $A$-boundaries.

(ii) $(G, \sigma)$ is *A-connected* if, for every $b \in Z(G, A)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$. That is, for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{h \in H_G(v)} \tau(h) f(e_h) = b(v).$$

**Remark.** 1. A signed graph $G$ with $\epsilon_N(G, \sigma) = 0$ can be switched to an ordinary graph, which allows us to study the nowhere-zero flow property and group connectivity property by analyzing its equivalent ordinary graph. In particular, we say a signed graph $G$ with $\epsilon_N(G, \sigma) = 0$ is $A$-connected if and only if its switch equivalent ordinary graph is $A$-connected.

2. It is obvious that a signed graph $(G, \sigma)$ with $E_\sigma = \{e_0\}$ does not admit a nowhere-zero integer-valued flow or a nowhere-zero $A$-flow if $|A|$ is odd.

For that reason, we only consider the group connectivity for 2-*unbalanced signed graphs*. It is also noted that in a 2-unbalanced signed graph, the sum of boundaries in Eq. (1) could be any element in $2A$, instead of zero for ordinary graphs.

## 2. Basic properties of group connectivity of signed graphs

In this section we present several basic properties on $A$-connectedness of signed graphs.

**Proposition 2.1.** *Each of the following holds.*

(a) *A-connectedness does not depend on the orientation.*

(b) *A-connectedness is invariant under switch operation.*

(c) *Let $G$ be a 2-unbalanced signed graph. If $|A|$ is even and $G$ is A-connected, then $G$ is connected. If $|A|$ is odd, then $G$ is A-connected if and only if each component of $G$ is A-connected.*

**Proof.** (a) is straightforward by the definition.

(b) Let $(G, \sigma)$ be a 2-unbalanced $A$-connected signed graph with orientation $\tau$. As every switching operation can be composed from the switching operations on trivial edge-cut, it is sufficient to verify ($b$) for the switch operation $\zeta$ on the trivial edge-cut $S = E_G(u)$ for any vertex $u$. Denote $\sigma' = \sigma\zeta$ the equivalent signature of $\sigma$. Let $\tau'$ be the orientation of $(G, \sigma')$ such that $\tau'(h) = -\tau(h)$ if $h \in H_G(u)$ and $\tau'(h) = \tau(h)$ otherwise. We are to show that $(G, \sigma')$ is $A$-connected.

Let $b' \in Z(G, A)$ be an $A$-boundary and define a mapping $b : V(G) \mapsto A$ to be $b(u) = -b'(u)$ and $b(v) = b'(v), \forall v \in V(G) \setminus \{u\}$. Since $\sum_{v \in V(G)} b'(v) \in 2A$, we have

$$\sum_{v \in V(G)} b(v) = -b'(u) + \sum_{v \in V(G) \setminus \{u\}} b'(v) = \sum_{v \in V(G)} b'(v) - 2b'(u) \in 2A.$$

Thus $b \in Z(G, A)$ is also an $A$-boundary of $(G, \sigma)$. Since $(G, \sigma)$ is $A$-connected, there exists a function $f \in F^*(G, A)$ such that, for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{h \in H_G(v)} \tau(h) f(e_h) = b(v).$$

By the setting of $\tau'$ in $(G, \sigma')$, we have $\partial f(v) = \sum_{h \in H_G(v)} \tau'(h) f(e_h) = b(v) = b'(v)$ for any vertex $v \in V(G) \setminus \{u\}$. In addition,

$$\partial f(u) = \sum_{h \in H_G(u)} \tau'(h) f(e_h) = \sum_{h \in H_G(u)} -\tau(h) f(e_h) = -b(u) = b'(u).$$

Therefore, $\partial f = b'$ in the signed graph $(G, \sigma')$ with orientation $\tau'$. Since $b'$ is arbitrary, $(G, \sigma')$ is $A$-connected.

(c) If $|A|$ is even, then there is an element $\beta \in A \setminus 2A$. Suppose that $G$ is not connected. Let $G_1$ be one component. Let $b \in Z(G, A)$ be an $A$-boundary function such that $\sum_{v\in V(G_1)} b(v) = \beta$, and $\sum_{v\in V(G)\setminus V(G_1)} b(v) = \beta$. Then there is no $f \in F^*(G, A)$ such that $\partial f = b$ by Eq. (1). Thus $G$ is connected.

If $|A|$ is odd, then $2A = A$. Hence every mapping $b : V(G) \mapsto A$ is an $A$-boundary. Thus it is easy to see that $G$ is $A$-connected if and only if each component of $G$ is $A$-connected. ■

By Proposition 2.1(c), we only discuss $A$-connectedness for connected signed graphs for convenience.

A connected base of a signed graph is a maximal spanning connected subgraph which contains neither balanced circuits nor barbells. In other words, a *connected base* $T$ of an unbalanced signed graph $(G, \sigma)$ is a spanning tree of its underlying ordinary graph plus an extra edge such that $T$ contains a unique unbalanced circuit. It plays the same role as spanning trees in ordinary graphs. The concept of bases is from signed graphic matroid introduced by Zaslavsky [19,20].

The following two propositions are originally proved for ordinary graphs in [5] and they can be extended to unbalanced signed graphs.

**Proposition 2.2.** *Let $(G, \sigma)$ be an unbalanced signed graph containing a connected base and $A$ be an abelian group. Let $\tau$ be an orientation of $(G, \sigma)$. Then, for each $b \in Z(G, A)$, there is a function $f \in F(G, A)$ such that $\partial f = b$.*

**Proof.** By the definition of $Z(G, A)$, Proposition 2.2 is preserved under switch operation. Hence it is sufficient to consider the case when $(G, \sigma)$ itself is a connected base with a unique negative edge. That is, $E^+_\sigma(G)$ induces a spanning tree. Let $e = v_1v_2$ be the unique negative edge in the unbalanced circuit of $(G, \sigma)$.

Let $b \in Z(G, A)$ with $\sum_{v\in V(G)} b(v) = 2\alpha$. Denote $G' = G - e$ and define $b' : V(G') \mapsto A$ by $b'(v_1) = b(v_1) - \alpha$, $b'(v_2) = b(v_2) - \alpha$ and $b'(v) = b(v)$ if $v \in V(G') \setminus \{v_1, v_2\}$. Then $b'$ is a zero sum boundary in the ordinary graph $G'$. Applying Proposition 2.1 of [5], there exists $f \in F(G', A)$ such that $\partial f = b'$ in $G'$. Extend $f$ to $E(G)$ by setting $f(e) = \alpha$ if $e$ is a sink edge, $f(e) = -\alpha$ if $e$ is a source edge. Then we have $\partial f = b$. ■

**Proposition 2.3.** *Let $(G, \sigma)$ be a connected 2-unbalanced signed graph with orientation $\tau$ and $A$ be an abelian group. Then the following statements are equivalent:*

(i) $(G, \sigma)$ *is $A$-connected.*

(ii) Given any $\bar{f} \in F(G, A)$, *there exists an $A$-flow $f$ such that $f(e) \neq \bar{f}(e)$ for every $e \in E(G)$.*

(iii) *Given two functions $\bar{f} \in F(G, A)$ and $b \in Z(G, A)$, there is a function $f \in F(G, A)$ which satisfies $\partial f = b$ and $f(e) \neq \bar{f}(e)$ for every $e \in E(G)$.*

**Proof.** The proof of Proposition 2.3 is a straightforward application of Proposition 2.2 and thus omitted. See [5] for a similar proof of this property in ordinary graphs. ■

For ordinary graphs, Jaeger et al. [5] pointed out that the monotonicity of group connectivity fails by presenting some graphs which are $\mathbb{Z}_5$-connected but not $\mathbb{Z}_6$-connected. It is unknown that whether $A_1$-connectedness implies $A_2$-connectedness for two nonisomorphic groups $A_1, A_2$ with $|A_1| = |A_2|$. It was even unknown for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ until very recently Hušek et al. [4] constructed two graphs and used a computer to verify that their $\mathbb{Z}_4$-connectedness and $\mathbb{Z}_2 \times \mathbb{Z}_2$-connectedness are distinct. For signed graphs we have the following proposition.

**Proposition 2.4.** *There are signed graphs that are $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected.*

**Proof.** Let $(G, \sigma)$ be the signed graph obtained from $K_2$ by adding one negative loop at each vertex. We will show that $(G, \sigma)$ is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected.

For $\mathbb{Z}_2 \times \mathbb{Z}_2$-connectedness, since $0 \in Z(G, \mathbb{Z}_2 \times \mathbb{Z}_2)$, set $b(v) = 0$ for each vertex $v$. Then there is no $f \in F^*(G, \mathbb{Z}_2 \times \mathbb{Z}_2)$ such that $\partial f = b$. Thus, $(G, \sigma)$ is not $\mathbb{Z}_2 \times \mathbb{Z}_2$-connected. It is easy to verify $(G, \sigma)$ is $\mathbb{Z}_4$-connected by checking all possible $\mathbb{Z}_4$-boundaries in $Z(G, \mathbb{Z}_4)$. ■

## 3. Contractible configuration and reduction

Let $\mathcal{P}$ be a signed graphic property. A signed graph $(H, \sigma)$ is a *contractible configuration of* $\mathcal{P}$ if, for every signed graph $(G, \sigma')$ containing $(H, \sigma)$ as a subgraph, $G/H$ has the property $\mathcal{P}$ if and only if $G$ has the property $\mathcal{P}$. For ordinary graphs, it is well-known that $A$-connected graphs are contractible configurations for nowhere-zero $A$-flow problems. The following theorem shows that *$A$-connected signed graphs are contractible configurations for nowhere-zero $A$-flow problems and $A$-connected problems of signed graphs.*

**Theorem 3.1.** *Let $A$ be an abelian group and let $(H, \sigma)$ be a signed graph. Assume that either $E^-_\sigma(H) = \emptyset$ and $H$ is $A$-connected as an ordinary graph or $(H, \sigma)$ is a 2-unbalanced $A$-connected signed graph. Then, for each 2-unbalanced signed graph $(G, \sigma')$ containing $(H, \sigma)$ as a subgraph, we have the following.*

*(a) $G$ admits a nowhere-zero $A$-flow if and only if $G/H$ admits a nowhere-zero $A$-flow;*

*(b) $G$ is $A$-connected if and only if $G/H$ is $A$-connected.*

**Proof.** We only prove (b) since the proof of (a) is similar to the proof of (b) by setting the $A$-boundary $b = 0$.

The necessity in (b) is obvious since the group connectivity is preserved under contraction. We now prove the sufficiency.

Let $\tau$ be an orientation of $(G, \sigma')$ and $b \in Z(G, A)$ be an $A$-boundary function. We still use $\tau$ to denote the corresponding orientation in $G/H$. Denote $v_H$ to be the vertex in $G/H$ which $H$ is contracted into. For convenience let $E_\sigma^-(H)$ denote the set of all negative edges of $(H, \sigma)$, as well as the set of negative loops incident with $v_H$ in $G/H$ obtained by contracting $H$. Define $b_1(v_H) = \sum_{v\in V(H)} b(v)$ and $b_1(v) = b(v)$ if $v \in V(G/H) \setminus \{v_H\}$. Then $b_1 \in Z(G/H, A)$. Since $G/H$ is $A$-connected, there exists $f_1 \in F^*(G/H, A)$ such that $\partial f_1 = b_1$. $(\tau, f_1)$ extends to $G$ such that $f_1$ inherits the corresponding value for any edge in $(E(G) - E(H)) \cup E_\sigma^-(H)$ and 0 otherwise.

For each vertex $v \in V(H)$, denote the set of half edges incident with $v$ in $E(G) - E(H)$ and in $E_\sigma^-(H)$ by $X_1(v)$ and $X_2(v)$, respectively. Define $b_2 : V(H) \mapsto A$ by

$$b_2(v) = b(v) - \sum_{h\in X_1(v)} \tau(h)f_1(e_h). \tag{2}$$

Since $\partial f_1 = b_1$ in $G/H$, we have

$$\sum_{v\in V(H)} \sum_{h\in X_1(v)\cup X_2(v)} \tau(h)f_1(e_h) = \partial f_1(v_H) = b_1(v_H) = \sum_{v\in V(H)} b(v).$$

Hence, by Eq. (2),

$$\begin{aligned}\sum_{v\in V(H)} b_2(v) &= \sum_{v\in V(H)} b(v) - \sum_{v\in V(H)} \sum_{h\in X_1(v)} \tau(h)f_1(e_h)\\ &= \sum_{v\in V(H)} \sum_{h\in X_2(v)} \tau(h)f_1(e_h)\\ &= \sum_{e\in E_\sigma^-(H)} \pm 2f_1(e) \in 2A.\end{aligned}$$

Thus $b_2 \in Z(H, A)$. In the case of $E_\sigma^-(H) = \emptyset$, $b_2$ is a zero sum function. Since $H$ is $A$-connected and by definition, there exists $f_2 \in F^*(H, A)$ such that $\partial f_2 = b_2$. Let $f_1'$ be the restriction of $f_1$ on $E(G) - E(H)$ and $f = f_1' + f_2$. Then, for each vertex $v \in V(H)$, it follows from Eq. (2) that

$$\begin{aligned}\partial f(v) &= \partial f_1'(v) + \partial f_2(v)\\ &= \sum_{h\in X_1(v)} \tau(h)f_1(e_h) + b_2(v)\\ &= \sum_{h\in X_1(v)} \tau(h)f_1(e_h) + [b(v) - \sum_{h\in X_1(v)} \tau(h)f_1(e_h)]\\ &= b(v).\end{aligned}$$

Therefore, $\partial f = b$ and $f \in F^*(G, A)$. By definition, $(G, \sigma')$ is $A$-connected. ■

Let $K_1^{-t}$ be the graph obtained from $K_1$ by adding $t$ negative loops. It is easy to see that $K_1^{-t}$ is $A$-connected for any abelian group of order $|A| \geq 3$ if $t \geq 2$. Theorem 3.1 leads to a reduction method for verifying $A$-connectedness of 2-unbalanced signed graphs, which is an extension of Catlin's reduction method on ordinary graphs (see [1,8]).

**Lemma 3.2.** *A 2-unbalanced signed graph $(G, \sigma)$ is $A$-connected if and only if it can be contracted to $K_1^{-t}$ for some integer $t \geq 2$ by contracting its $A$-connected subgraph recursively.*

The following lemma follows immediately as an application of Theorem 3.1(b) and Lemma 3.2.

**Lemma 3.3.** *Let $(G, \sigma)$ be a 2-unbalanced signed graph.*

(i) *If $G[E_\sigma^+]$ is spanning and $A$-connected as an ordinary graph, then $(G, \sigma)$ is $A$-connected.*

(ii) *Suppose that $G$ is 2-edge-connected. If $G - v$ is a 2-unbalanced $A$-connected signed graph, then $(G, \sigma)$ is $A$-connected.*

These methods will be applied in the next two sections to verify group connectivity of various signed graphs.

## 4. Group connectivity of highly connected signed graphs

### *4.1. A-connectedness for $|A| \geq 4$*

Jaeger et al. [5] investigated the relation of edge-connectivity and group connectivity. They showed that every 4-edge-connected graph is $A$-connected for any abelian group of order $|A| \geq 4$ and every 3-edge-connected graph is $A$-connected for $|A| \geq 6$. We obtain analogous results for signed graphs with slightly higher edge-connectivity.

**Theorem 4.1.** *Let $G$ be a 2-unbalanced signed graph.*
(i) *If $G$ is 4-edge-connected, then $G$ is $A$-connected for any abelian group $A$ with order $|A| = 4$ or $|A| \geq 6$.*
(ii) *If $G$ is 6-edge-connected, then $G$ is $A$-connected for any abelian group $A$ with order $|A| \geq 4$.*

It is unknown whether every 4-edge-connected 2-unbalanced signed graph is $\mathbb{Z}_5$-connected or not.

Theorem 4.1 is a corollary of Theorem 4.3 below, together with a theorem of Raspaud and Zhu [13] on the number of edge disjoint connected bases in highly edge-connected signed graphs.

**Theorem 4.2** (*Raspaud and Zhu [13]*). *Let $(G, \sigma)$ be a $k$-unbalanced signed graph. If $G$ is $2k$-edge-connected, then $(G, \sigma)$ has $k$ edge disjoint connected bases.*

**Theorem 4.3.** *Let $(G, \sigma)$ be a 2-unbalanced signed graph with orientation $\tau$.*
(i) *If $G$ has two edge disjoint connected bases, then $(G, \sigma)$ is $A$-connected for any abelian group $A$ with order $|A| = 4$ or $|A| \geq 6$.*
(ii) *If $G$ contains three edge disjoint connected bases, then $(G, \sigma)$ is $A$-connected for $|A| \geq 4$.*

To prove Theorem 4.3, we also need the following theorem due to Cheng et al. [2], which extends a $\mathbb{Z}_2$-flow to an integer-valued 3-flow by adding an appropriate $T$-join to connect the support of $\mathbb{Z}_2$-flow.

**Theorem 4.4** (*Cheng et al. [2]*). *If a signed graph $(G, \sigma)$ is connected and admits a $\mathbb{Z}_2$-flow $f_1$ such that $supp(f_1) = \{e : f_1(e) \neq 0\}$ contains an even number of negative edges, then it also admits an integer-valued 3-flow $f_2$ with $supp(f_1) = \{e \in E(G) : f_2(e) = \pm 1\}$.*

A graph (signed graph) is *even* if the degree of each vertex is even. A graph (signed graph) is *eulerian* if it is even and connected.

**Theorem 4.5** (*Xu and Zhang [18]*). *A connected signed graph $(G, \sigma)$ admits a nowhere-zero 2-flow if and only if it is eulerian and contains even number of negative edges.*

Now we are ready to prove Theorem 4.3.

**Proof of Theorem 4.3.** (i) Let $A$ be an abelian group with order $|A| = 4$ or $|A| \geq 6$. Let $b \in Z(G, A)$. We will show that there is a function $f \in F^*(G, A)$ such that $\partial f = b$. Let $B_1$ and $B_2$ be two edge disjoint connected bases of $(G, \sigma)$. Since $(G, \sigma)$ is unbalanced, each $B_i$ is a spanning tree (of ordinary graph) plus one additional edge to make a unique unbalanced circuit.

Pick $x \in A^*$. Let $f_2 \in F(G, A)$ with $f_2(e) = x$ if $e \in E(G) - B_1$ and $f_2(e) = 0$ if $e \in B_1$. Then $\sum_{v \in V(G)} \partial f_2(v) = 2kx \in 2A$ for some integer $k$, and so $-\partial f_2 + b \in Z(G, A)$. By Proposition 2.2, there is a function $f_1 \in F(B_1, A)$ such that $\partial f_1 = -\partial f_2 + b$.

Denote $E' = \{e \in B_1 | f_1(e) = 0\}$. For each $e \in E'$, $B_2 + e$ contains a unique signed circuit, which is either a balanced circuit $C_e$ of $(G, \sigma)$, or a barbell of $(G, \sigma)$, consisting of two edge disjoint unbalanced circuits $C_e^1, C_e^2$ and a path $P_e$ (possibly of length 0) connecting the two circuits. Clearly, $e \notin E(P_e)$ if the signed circuit is a barbell. In the former case, let $C(e) = C_e$; in the later case, let $C(e) = C_e^1 \cup C_e^2$. In any case, $C(e)$ is an even subgraph with even number of negative edges, i.e. $\sigma(C(e)) = 1$.

Let $G' = \triangle_{e \in E'} C(e)$. Then $G'$ is an even subgraph of $H = G[B_2 \cup E']$ and thus admits a nowhere-zero $\mathbb{Z}_2$-flow. Since $\sigma(G') = \prod_{e \in E'} \sigma(C(e)) = 1$, $G'$ contains an even number of negative edges. Since $H$ is connected, by Theorem 4.4, $H$ admits a 3-flow $f_3$ such that $|f_3(e)| = 1$ if and only if $e \in E(G')$.

Pick $y \in A^*$ such that $y \neq \pm x$ and $2y \neq \pm x$. We first show that such $y$ does exist. Obviously if $|A| \geq 6$ or $x = -x$, such $y$ exist. If $|A| = 4$ and $x \neq -x$, then $A \cong \mathbb{Z}_4$, $x = 1$ or 3, and thus $2a \notin \{x, -x\}$ for every element $a \in A^*$. Thus in each case, such $y$ does exist.

Let $f = f_1 + f_2 + yf_3$. Then $f(e) \in \{x, \pm y, x \pm y, x \pm 2y\} \subseteq A^*$. Thus $f \in F^*(G, A)$. Moreover, $\partial f = \partial f_1 + \partial f_2 = b$. Therefore $(G, \sigma)$ is $A$-connected.

(ii) The argument is very similar to that of (i). Because of the connected base $B_3$, the graph $G'$ is connected and thus by Theorem 4.5, it admits a nowhere-zero 2-flow. This would eliminate the constraint $2y \neq \pm x$ in the proof of (i). In the following we give some details to show how to find such a connected graph $G'$.

Let $B_1, B_2, B_3$ be three edge disjoint connected bases. Pick $x \in A^*$. We first define $f_2 \in F(G, A)$ such that $f_2(e) = x$ if $e \in E(G) - B_1 - B_3$ and $f_2(e) = 0$ otherwise. By Proposition 2.2, there is a function $f_1 \in F(B_1, A)$ such that $\partial f_1 = -\partial f_2 + b$. Denote $E' = \{e \in B_1 | f_1(e) = 0\} \cup B_3$. Then $G' = \triangle_{e \in E'} C(e)$ is connected since it contains the connected base $B_3$. ■

Note that, in Theorem 4.1(ii), if $\epsilon_N = 2$, then $(G, \sigma)$ does not contain three edge disjoint bases, but this case is easy. We may switch $(G, \sigma)$ to an equivalent signed graph $(G', \sigma')$ with $|E^-_{\sigma'}(G')| = 2$. Then $E^+_{\sigma'}$ induces a 4-edge-connected ordinary graph, which is $A$-connected for any $|A| \geq 4$ by a theorem of Jaeger et al. [5]. By Lemma 3.3(i), $(G', \sigma')$ is $A$-connected for any $|A| \geq 4$, and so does $(G, \sigma)$ by Proposition 2.1(b).

### 4.2. $\mathbb{Z}_3$-connectedness

$\mathbb{Z}_3$-connectedness of ordinary graphs has been studied extensively (see [8,9]). The following is a basic property extended from ordinary graphs to signed graphs, whose proof is straightforward by definition (see [8] for a similar proof for ordinary graphs).

**Proposition 4.6.** *Let $(G, \sigma)$ be a 2-unbalanced signed graph. The following are equivalent.*
(i) *$(G, \sigma)$ is $\mathbb{Z}_3$-connected.*
(ii) *For any $b \in Z(G, \mathbb{Z}_3)$, there exists an orientation $\tau$ such that, for every vertex $v \in V(G)$,*

$$d_\tau^+(v) - d_\tau^-(v) \equiv b(v) \pmod 3.$$

(iii) *For any $b \in Z(G, \mathbb{Z}_3)$, there exists an orientation $\tau$ such that $d_\tau^+(v) \equiv b(v) \pmod 3$ for every vertex $v \in V(G)$.*

The following proposition characterizes $\mathbb{Z}_3$-connected signed graphs with exactly two negative edges.

**Proposition 4.7.** *Let $(G, \sigma)$ be a 2-unbalanced signed graph with $E_\sigma^- = \{e_1, e_2\}$. Then $(G, \sigma)$ is $\mathbb{Z}_3$-connected if and only if $G - e_1 - e_2$ is $\mathbb{Z}_3$-connected (as an ordinary graph).*

**Proof.** "$\Leftarrow$" follows from Lemma 3.3(i). We are to show "$\Rightarrow$".

For every vertex $v \in V(G)$, denote $c(v)$ to be the number of half edges in $E_\sigma^-$ incident with $v$. Then we have $\sum_{v\in V(G)} c(v) = 4$. Note that $c(v) \in \{0, 1, 2, 3, 4\}$ since negative loops are allowed in $(G, \sigma)$.

Let $G' = G - e_1 - e_2$. For any zero sum boundary function $b'$ of $G'$, we will show there exists an orientation $D'$ of $G'$ such that $d_{D'}^+(v) - d_{D'}^-(v) \equiv b'(v) \pmod 3$ for every vertex $v \in V(G')$. Set $b(v) = b'(v) - c(v)$ for every vertex $v \in V(G)$. Then $b \in Z(G, \mathbb{Z}_3)$ and

$$\sum_{v\in V(G)} b(v) \equiv \sum_{v\in V(G')} b'(v) - \sum_{v\in V(G)} c(v) \equiv -1 \pmod 3.$$

Since $(G, \sigma)$ is $\mathbb{Z}_3$-connected and by Proposition 4.6(ii), there exists an orientation $\tau$ of $(G, \sigma)$ such that $d_\tau^+(v) - d_\tau^-(v) \equiv b(v) \pmod 3$ for every vertex $v \in V(G)$. It follows that $\sum_{v\in V(G)}(d_\tau^+(v) - d_\tau^-(v)) \equiv -1 \pmod 3$. Note that every sink edge contributes 2 and every source edge contributes $-2$, while each positive edge contributes zero to the sum $\sum_{v\in V(G)}(d_\tau^+(v) - d_\tau^-(v))$. Thus, $e_1$ and $e_2$ are both oriented as source edges. Let $D'$ be the restriction of $\tau$ on $E_\sigma^+$. Then, for every $v \in V(G')$,

$$d_{D'}^+(v) - d_{D'}^-(v) = d_\tau^+(v) - d_\tau^-(v) + c(v) \equiv b'(v) \pmod 3.$$

Hence $D'$ is the orientation as desired. ■

The proof of the theorem of Lovàsz et al. [10] shows every graph obtained from 6-edge-connected graph deleting three edges is still $\mathbb{Z}_3$-connected (see [17]). Therefore, we obtain the following result by Lemma 3.3(i).

**Corollary 4.8.** *Every 6-edge-connected signed graph with $\epsilon_N \in \{0, 2, 3\}$ is $\mathbb{Z}_3$-connected.*

The following theorem was proved by Zhu [22] for 3-unbalanced signed graph. In fact, we show it holds for all 2-unbalanced signed graphs as a corollary of Corollary 4.8 and Lemma 3.3(i).

**Theorem 4.9.** *Every 11-edge-connected 2-unbalanced signed graph is $\mathbb{Z}_3$-connected.*

**Proof.** By Proposition 2.1(b) that $A$-connectedness is an invariant under switch operation, we may assume that $|E_\sigma^-(G)| = \epsilon_N$. Since $(G, \sigma)$ is a 11-edge-connected signed graph with minimal number of negative edges in the switch equivalent class, $|S \cap E_\sigma^-(G)| \le \frac{|S|}{2}$ for each edge-cut $S$. Therefore $E_\sigma^+$ is 6-edge-connected and hence is $\mathbb{Z}_3$-connected by Corollary 4.8. By Lemma 3.3(i), $G$ is $\mathbb{Z}_3$-connected. ■

It would be interesting to reduce the edge-connectivity condition. We believe 6-edge-connectivity (or even 5) should be able to guarantee $\mathbb{Z}_3$-connectedness for signed graphs.

**Conjecture 4.10.** *Every 5-edge-connected 2-unbalanced signed graph is $\mathbb{Z}_3$-connected.*

In particular, Conjecture 4.10, if true, would imply the conjecture of Jaeger et al. that *every 5-edge-connected ordinary graph is $\mathbb{Z}_3$-connected* by Proposition 4.7, and thus implies Tutte's 3-Flow Conjecture by Kochol's result in [7]. The next section verifies Conjecture 4.10 for some families of signed graphs.

## 5. Group connectivity of some families of signed graphs

In this section, we study group connectivity of signed $K_4$-minor free graphs and signed complete graphs. Specifically, by applying the reduction method introduced in Section 3, we will verify Conjecture 4.10 for 5-edge-connected signed $K_4$-minor free graphs, signed complete graphs, and signed $k$-trees with $k \ge 5$.

**Theorem 5.1.** *Every 5-edge-connected 2-unbalanced signed $K_4$-minor free graph is $A$-connected for any abelian group $A$ with $|A| \ge 3$.*

**Theorem 5.2.** *Every 2-unbalanced signed $K_n$ with $n \ge 6$ is $A$-connected for any abelian group $A$ with $|A| \ge 3$.*

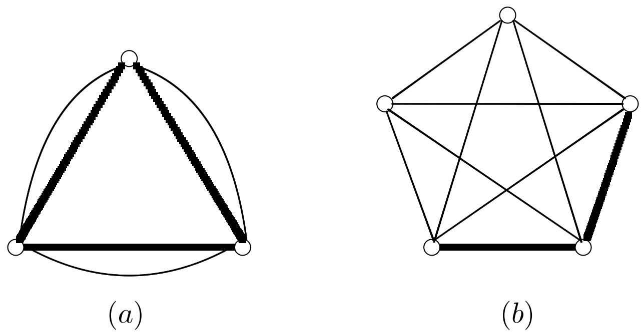


**Fig. 1.** Non $\mathbb{Z}_3$-connected signed series–parallel graph and signed $K_5$.

An ordinary graph $G$ is *series–parallel* if it can be obtained from $K_2$ by a sequence of series and parallel extensions. Signed series–parallel graphs are obtained from ordinary series–parallel graphs by assigning signatures. Kaiser and Rollová [6] proved that every 2-unbalanced signed series–parallel graph admits a nowhere-zero 6-flow provided that it has a nowhere-zero integer flow. It is known that a series–parallel graph is $K_4$-minor free. Thus by Theorem 5.1, we have the following. By the result of Xu and Zhang [18] on the equivalence of $\mathbb{Z}_3$-flow and integer 3-flow on 2-edge-connected signed graphs, Theorem 5.1 strengthens Kaiser and Rollová's 6-flow result to 3-flow if the edge connectivity increases to 5.

**Theorem 5.3.** *Every* 5*-edge-connected* 2*-unbalanced signed series–parallel graph admits a nowhere-zero* 3*-flow.*

Similarly Theorem 5.2 implies the following result by Máčajová and Rollová [12] on signed complete graph.

**Theorem 5.4.** *Every* 2*-unbalanced signed* $K_n$ *with* $n \geq 6$ *admits a nowhere-zero* 3*-flow.*

Let $k \geq 1$ be an integer. A graph on $n$ vertices is called a *k-tree* if either it is a clique with order $n = k + 1$, or it is obtained from a $k$-tree $T_{n-1}$ on $n - 1$ vertices by adding a new vertex which is adjacent to a $k$-clique of $T_{n-1}$, and is non-adjacent to any of the other vertices of $T_{n-1}$. A signed $k$-tree is obtained from an ordinary $k$-tree by assigning signatures.

The following is an immediate corollary of Theorem 5.2 together with Lemma 3.3, which verifies Conjecture 4.10 for signed $k$-trees.

**Corollary 5.5.** *Every* 2*-unbalanced signed k-tree with* $k \geq 5$ *is* $\mathbb{Z}_3$*-connected and thus admits a nowhere-zero* 3*-flow.*

**Remark.** 1. In Theorem 5.1, the 5-edge-connectivity cannot be reduced to 4-edge-connectivity. Wu et al. [17] constructed a 4-edge-connected 2-unbalanced signed $K_4$-minor free graph which does not admit a nowhere-zero 3-flow (and hence is not $\mathbb{Z}_3$-connected). See Fig. 1(a), where the thick lines represent negative edges.

2. It is proved by Lai et al. (see [8]) that every ordinary complete graph with at least 5 vertices is $\mathbb{Z}_3$-connected. However not every 2-unbalanced signed $K_5$ is $\mathbb{Z}_3$-connected. For example, the signed graph in Fig. 1(b) (the thick lines represents negative edges) is not $\mathbb{Z}_3$-connected by Proposition 4.7.

### *5.1. Proofs of Theorems 5.1 and 5.2*

The following lemma will be used in the proofs of Theorems 5.1 and 5.2.

**Lemma 5.6.** (i) [5] *The circuit* $C_n$ *of length n is A-connected if and only if* $n + 1 \leq |A|$.
(ii) [11] *The wheel* $W_4$ *and the graph* $G_1$ *on* 6 *vertices in Fig. 2 are* $\mathbb{Z}_3$*-connected.*
(iii) [3] *Every* $K_4$*-minor free simple graph has a vertex of degree at most* 2.

Now we are ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** Let $(G, \sigma)$ be a counterexample with $V(G)$ minimum. We may assume $|E_\sigma^-(G)| = \epsilon_N$. Clearly, any $K_1^{-t}$ is $A$-connected for $|A| \geq 3$ and $t \geq 2$. So $|V(G)| \geq 2$ and $|E_\sigma^+| > 0$.

Denote $G_0$ to be the underlying ordinary simple graph of $G$. Since $G_0$ contains no $K_4$-minor, by Lemma 5.6(iii), $G_0$ contains a vertex of degree at most 2, say $v$. Since $G$ is 5-edge-connected, there are at least three positive edges incident with $v$. Thus there is a digon $C_2$ with two positive edges containing $v$. By Lemma 5.6, the digon $C_2$ is $A$-connected for $|A| \geq 3$. By the minimality of $(G, \sigma)$, $G/vv_1$ is $A$-connected for $|A| \geq 3$. Hence $(G, \sigma)$ is $A$-connected for $|A| \geq 3$ by Theorem 3.1. ■

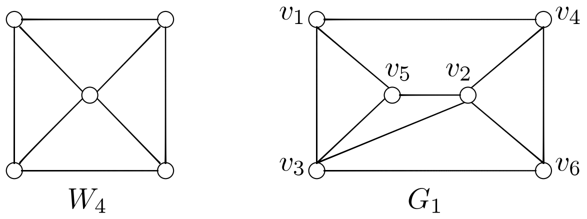


**Fig. 2.** The graphs $W_4$ and $G_1$.

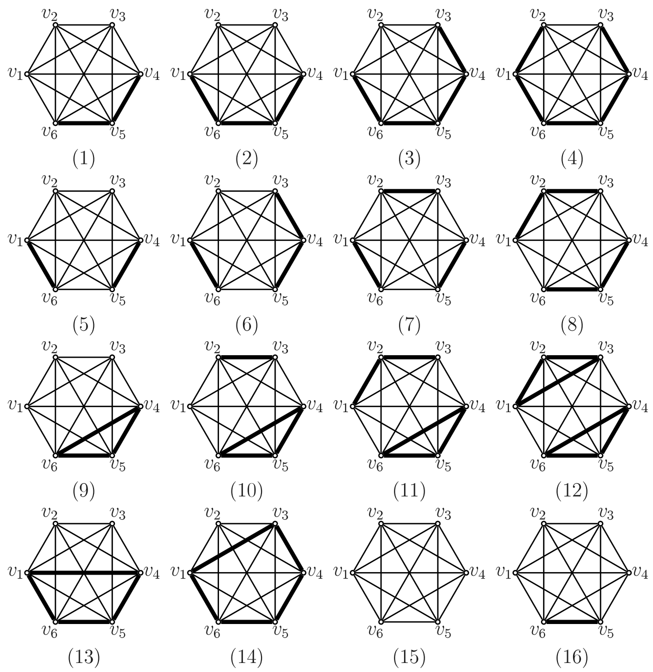


**Fig. 3.** 16 nonisomorphic signed complete graphs on 6 vertices.

Let $(G, \sigma)$ be a signed graph and $uv_1, uv_2 \in E(G)$. $(G_{[u,v_1v_2]}, \sigma')$ is the signed graph obtained from $(G, \sigma)$ by deleting $uv_1, uv_2$, keeping the sign of other edges and adding a new edge $v_1v_2$ with sign $\sigma'(v_1v_2) = \sigma(uv_1)\sigma(uv_2)$.

The lifting lemma below follows easily from the definition, and thus the proof is omitted.

**Lemma 5.7** (*Lifting*)**.** *Let $(G, \sigma)$ be a 2-unbalanced signed graph and $uv_1, uv_2 \in E(G)$. If $(G_{[u,v_1v_2]}, \sigma')$ is a 2-unbalanced A-connected signed graph, then $(G, \sigma)$ is A-connected.*

Máčajová and Rollová classified all nonisomorphic signed $K_6$ in [12].

**Lemma 5.8** (*Máčajová and Rollová [12]*)**.** *There are 16 nonisomorphic signed complete graphs on 6 vertices, depicted in Fig. 3.*

**Proposition 5.9.** *Every 2-unbalanced signed $K_6$ is A-connected for any A with $|A| \geq 3$.*

**Proof.** Since a circuit of length 2 or 3 is an $A$-connected ordinary graph for $|A| \geq 4$ by Lemma 5.6(i), it is easy to check $E_\sigma^+$ is $A$-connected for $|A| \geq 4$ for each of them. By Lemma 3.3(i), every 2-unbalanced signed $K_6$ is $A$-connected for $|A| \geq 4$. We are to verify $\mathbb{Z}_3$-connectedness below.

(1)–(10) are $\mathbb{Z}_3$-connected since $E_\sigma^+$ is $\mathbb{Z}_3$-connected and spanning and by Lemma 3.3(i). In particular, for $E_\sigma^+$ in (4) or (8), it is either isomorphic to $G_1$ or contains $G_1$ as a spanning subgraph. By Lemma 5.6(ii), $G_1$ is $\mathbb{Z}_3$-connected, and so (4) and (8) are $\mathbb{Z}_3$-connected by Lemma 3.3(i).

For the rest, we apply lifting lemma to show $\mathbb{Z}_3$-connectedness by lifting two negative edges to obtain an extra positive edge. For (11), lift $v_1v_2$, $v_2v_3$ to obtain a graph $G' = G_{[v_2,v_1v_3]}$. By Lemmas 3.2 and 5.6(i), $E'^+_{\sigma'}$ is $\mathbb{Z}_3$-connected by contracting 2-cycles consecutively. So, by Lemmas 3.3(i) and 5.7, (11) is $\mathbb{Z}_3$-connected.

For (12), lift $v_1v_2$, $v_2v_3$ to obtain a graph $G' = G_{[v_2,v_1v_3]}$, and then lift $v_4v_5$, $v_5v_6$ to obtain a graph $G'' = G'_{[v_5,v_4v_6]}$. Then $G''$ is $\mathbb{Z}_3$-connected with $\epsilon_N = 2$ since $E''^{+}_{\sigma''}$ is isomorphic to $G_1$. Hence (12) is $\mathbb{Z}_3$-connected by Lemmas 3.3(i) and 5.7.

For (13) and (14), lift $v_1v_6$, $v_5v_6$ to obtain a graph $G' = G_{[v_6,v_1v_5]}$. By Lemmas 3.2 and 5.6(i), $E'^{+}_{\sigma'}$ is $\mathbb{Z}_3$-connected by consecutively contracting 2-cycles. Then $G'$ is a 2-unbalanced $\mathbb{Z}_3$-connected signed graph, and so (13) and (14) are $\mathbb{Z}_3$-connected by Lemma 5.7. This completes the proof. ■

**Proof of Theorem 5.2.** Prove by induction on $n$. It is true for $n = 6$ by Proposition 5.9. Assume $n \geq 7$ and the statement is true for any positive integers smaller than $n$. Let $(G, \sigma)$ be a 2-unbalanced signed $K_n$. We may further assume $\epsilon_N(G, \sigma) = |E^-_\sigma|$ as $A$-connectedness is invariant under switch operation by Proposition 2.1(b). If $\epsilon_N(G, \sigma) = 2$, then $(G, \sigma)$ is $A$-connected for $|A| \geq 3$ as $E^+_\sigma$ is $A$-connected and by Lemma 3.3(i). Otherwise, assume $\epsilon_N(G, \sigma) \geq 3$. Let $v$ be a vertex in $G$ such that the number of negative edges incident with $v$ minimum. Then $G - v$ is 2-unbalanced. So $G - v$ is $A$-connected for $|A| \geq 3$ by induction hypothesis. Therefore, by applying Lemma 3.3(ii), $(G, \sigma)$ is $A$-connected for $|A| \geq 3$. The proof is completed. ■

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