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Vertex-coloring 3-edge-weighting of some graphs

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ABSTRACT

Let G be a non-trivial graph and $k \in \mathbb{Z}^+$. A vertex-coloring k-edge-weighting is an assignment $f : E(G) \to \{1, \ldots, k\}$ such that the induced labeling $f : V(G) \to \mathbb{Z}^+$, where $f(v) = \sum_{e \in E(v)} f(e)$ is a proper vertex coloring of G. It is proved in this paper that *every 4-edge-connected graph with chromatic number at most 4 admits a vertex-coloring 3-edge-weighting.*

1. Introduction

For technical reasons, all graphs considered in this paper are connected multigraphs with parallel edges but no loop. A graph with two vertices and m parallel edges is denoted by mK_2.

Let G be a graph. A vertex-coloring k-edge-weighting of G is an assignment $f : E(G) \to \{1, \ldots, k\}$ such that the induced labeling $f : V(G) \to \mathbb{Z}^+$, where $f(v) = \sum_{e \in E(v)} f(e)$, is a proper vertex coloring of G (see [1,2,3,6,11], or a comprehensive survey paper [9]).

In [6], Karoński, Luczak and Thomason conjectured *(the 1-2-3-conjecture) that every graph other than mK_2 admits a vertex coloring 3-edge-weighting.* It is proved in [5] that every graph other than mK_2 admits a vertex-coloring 5-edge-weighting. It also proved in [6] that every 3-colorable graph other than mK_2 admits a vertex-coloring 3-edge-weighting; and in [7] that every 4-colorable graph other than mK_2 admits a vertex-coloring 4-edge-weighting. In this paper, we extend some of these results by verifying the 1-2-3-conjecture for some graphs G with $\chi(G) \le 4$.

Theorem 1.1. *Every 4-edge-connected 4-colorable multigraph G admits a vertex-coloring 3-edge-weighting.*

1.1. Notation and terminology

We follow [4] and [12] for terms and notation.

A circuit is a connected 2-regular graph.

Let H_1 and H_2 be two subgraphs of a graph G. The symmetric difference of H_1 and H_2, denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$.

Let G be a graph. The set of odd vertices of G is denoted by $O(G)$. Let U be a subset of $V(G)$ with even order. A spanning subgraph Q is called T-join of G (with respected to U) if $O(Q) = U$.

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2. Proof of the main theorem

2.1. Sketch of an outline of the proof

Let $\beta : V(G) \to Z_4$ be a 4-coloring of G. We are to find a vertex-coloring 3-edge-weighting f such that $f(v) \equiv \beta(v) \pmod 4$ for every vertex v.

A necessary condition of β is $\sum_{v\in V(G)}\beta(v) \equiv 0 \pmod 2$ since

$$\sum_{v\in V(G)} f(v) \equiv 2 \sum_{e\in E(G)} f(e) \equiv 0 \pmod 2.$$

Let

$$W_\mu = \{v \in V(G) : \beta(v) - d_G(v) \equiv \mu \pmod 4\}.$$

The first step of the proof is to find a T-join Q with $O(Q) = W_1 \cup W_3$. Define $g : E(Q) \to \{2\}$, and $\beta' : V(G) \to Z_4$ such that

$$\beta'(x) \equiv \left\{\begin{matrix} \beta(x) - 2 & \text{if } x \in W_1 \cup W_3 \\ \beta(x) & \text{otherwise} \end{matrix}\right\} \pmod 4.$$

It will be proved that, in the subgraph $G - E(Q)$, $d_{G-E(Q)}(v) \equiv \beta'(v)$ or $\beta'(v) + 2 \pmod 4$ for every $v \in V(G)$.

In the second step, Lemma 2.2 is applied to find another edge-weight $f_0 : E(G) - E(Q) \to \{1, 3\}$ such that, for every $v \in V(G)$,

$$\beta'(v) \equiv \sum_{e\in E(v)-E(Q)} f_0(e) \pmod 4.$$

Thus, the combination of g and f_0 yields a vertex-coloring 3-edge-weighting of G.

By Tutte and Nash-Williams Theorem [8,10], a 4-edge-connected graph contains a pair of edge-disjoint spanning trees T_1, T_2. The subset Q is to be found in $G - T_2$, and the weight f_0 is assigned in $E(G) - E(Q)$. We notice that a straightforward application of Tutte–Nash-Williams Theorem is not sufficient due to a parity requirement for $|Q|$. Thus, Tutte–Nash-Williams Theorem is extended in Lemma 2.1 in order to meet the requirements of Lemma 2.2 in the second step of the proof.

2.2. Lemmas

Lemma 2.1. *If G is a 4-edge-connected non-bipartite graph, then $E(G)$ has a partition $\{T_1, T_2, F\}$ such that each T_i is a spanning tree and $T_1 + F$ contains an odd-circuit.*

Lemma 2.2. *Let H be a graph and let $\beta_H : V(H) \to Z_4$ be a mapping. Assume that*

(i) *H is connected;*

(ii) $\beta_H(v) \equiv d_H(v) \pmod 2$ *for each vertex $v \in V(H)$;*

(iii) $\sum_{v\in V(H)}\beta_H(v) \equiv 2|E(H)| \pmod 4$.

Then there exists a mapping $f_H : E(H) \to \{1, 3\}$ such that for each vertex $x \in V(H)$,

$$f_H(x) = \sum_{e\in E(x)} f_H(e) \equiv \beta_H(x) \pmod 4. \tag{1}$$

See Section 3 for proofs of both lemmas.

2.3. Proof of Theorem 1.1

We pay only attention to graphs with chromatic number $\chi = 4$ since it was proved in [6] that every multigraph G with $\chi(G) \le 3$ admits a vertex-coloring 3-edge-weighting.

I. Since $\chi(G) = 4$, there exists a vertex partition $\{V_0, V_1, V_2, V_3\}$ of $V(G)$ such that each V_i is an independent set, $i = 0, 1,$ 2 and 3. Renaming them if necessary, we can assume that $|V_1| + |V_3|$ is even. Define $\beta : V(G) \to \{0, 1, 2, 3\}$ such that $\beta(v) = i$ if $v \in V_i$. Then

$$\sum_{x\in V(G)} \beta(x) = |V_1| + 2|V_2| + 3|V_3| \equiv |V_1| + |V_3| \equiv 0 \pmod 2. \tag{2}$$

Our goal is to find an edge-weighting $f : E(G) \to \{1, 2, 3\}$ such that

$$\sum_{e\in E(x)} f(e) \equiv \beta(x) \pmod 4 \tag{3}$$

for every vertex x.

II. By Lemma 2.1, $E(G)$ has a partition $\{T_1, T_2, F\}$ where each T_i is a spanning tree and $F + T_1$ contains an odd circuit C_e.

III. Let

$$U = \{x \in V(G) | \ d_G(x) \equiv \beta(x) + 1 \pmod 2\}.$$

We have that

$$|U| \equiv 0 \pmod 2 \tag{4}$$

since, by (2),

$$0 \equiv 2|E(G)| = \sum_{x \in V(G)} d_G(x) \equiv |U| + \sum_{x \in V(G)} \beta(x) \equiv |U| \pmod 2.$$

Let Q_1 be a T-join with $O(Q_1) = U$ contained in the spanning tree T_1, and let $Q_2 = Q_1 \triangle C_e$, which is also a T-join with $O(Q_2) = U$ and is contained in $T_1 + F$.

IV. For each $i = 1, 2$, define $g_i : E(G) \to \{0, 2\}$ such that

$$g_i(e) = \begin{cases} 2 & e \in E(Q_i) \\ 0 & e \notin E(Q_i). \end{cases} \tag{5}$$

Let $G_i = G \setminus E(Q_i)$

In the remaining part of the paper, we are to find a mapping $f_i : E(G_i) \to \{1, 3\}$ for $i \in \{1, 2\}$ such that either $f_1 + g_1$ or $f_2 + g_2$ is a vertex coloring 3-edge-weighting of G. In order to apply Lemma 2.2 here, three conditions of the lemma will be verified one-by-one in the next subsection.

V. Let $\beta' : V(G) \to Z_4$ such that

$$\beta'(x) \equiv \left\{ \begin{matrix} \beta(x) - 2 & \text{if } x \in U \\ \beta(x) & \text{if } x \in V(G) \setminus U \end{matrix} \right\} \pmod 4. \tag{6}$$

Since $E(Q_i) \subseteq E(T_1 \cup C_e)$, for each $i = 1, 2$, the subgraph $G_i = G \setminus E(Q_i)$ contains the spanning tree T_2. So G_i is connected and satisfies Condition (i) of Lemma 2.2.

Then

$$\beta'(x) \equiv \left\{ \begin{matrix} \beta(x) - 2 \equiv d_G(x) - 1 \equiv d_{G_i}(x) & \text{if } x \in U \\ \beta(x) \equiv d_G(x) \equiv d_{G_i}(x) & \text{if } x \in V(G) \setminus U \end{matrix} \right\} \pmod 4.$$

It is easy to see that, for each $i = 1, 2$, G_i and β' satisfy Condition (ii) of Lemma 2.2.

$V(G_1) = V(G_2) = V(G)$. By (6) and (2) we have that, for each $i = 1, 2$,

$$\sum_{x \in V(G_i) = V(G)} \beta'(x) \equiv \sum_{x \in V(G) = V(G)} \beta(x) \equiv 0 \pmod 2. \tag{7}$$

Furthermore, $|E(Q_1)| + |E(Q_2)|$ is odd since C_e is a circuit of odd length and

$$|E(Q_1)| + |E(Q_2)| \equiv |E(Q_1 \triangle Q_2)| = |E(C_e)| \equiv 1 \pmod 2.$$

Hence,

$$\begin{aligned} |E(G_1)| + |E(G_2)| &= (|E(G)| - |E(Q_1)|) + (|E(G)| - |E(Q_2)|) \\ &= 2|E(G)| - (|E(Q_1)| + |E(Q_2)|) \end{aligned}$$

and is also odd.

By (7), we must have either $\sum_{x \in V(G_1)} \beta'(x) \equiv 2|E(G_1)| \pmod 4$ or $\sum_{x \in V(G_2)} \beta'(x) \equiv 2|E(G_2)| \pmod 4$.

Without loss of generality, suppose

$$\sum_{x \in V(G_1)} \beta'(x) \equiv 2|E(G_1)| \pmod 4,$$

which satisfies Condition (iii) of Lemma 2.2, and, therefore, by Lemma 2.2, there is a mapping $f_1 : E(G_1) \to \{1, 3\}$ such that

$$f_1(x) = \sum_{e \in E(x) \cap E(G_1)} f_1(e) \equiv \beta'(x) \pmod 4 \tag{8}$$

for each vertex $x \in V(G_1) = V(G)$.

VI. Let $f : E(G) \to \{1, 2, 3\}$ such that

$$f(e) = \begin{cases} g_1(e) & \text{if } e \in E(Q_1) \\ f_1(e) & \text{if } e \in E(G) \setminus E(Q_1) = E(G_1). \end{cases}$$

By Eqs. (8), (5) and (6), it is not difficult to verify that

$$f(x) = \sum_{e\in E(x)} f(e) \equiv f_1(x) + g_1(x) \equiv \begin{Bmatrix} \beta'(x) & \text{if } x \notin U \\ \beta'(x)+2 & \text{if } x \in U \end{Bmatrix} \equiv \beta(x) \pmod 4.$$

Therefore, f is a vertex-coloring 3-edge-weighting of G (satisfying Eq. (3)). □

3. Proofs of Lemmas 2.1 and 2.2

Proof of Lemma 2.2. Let $f : E(G) \to \{1, 3\}$ be an arbitrary mapping such that, for each vertex $x \in V(G)$, $f(x) = \sum_{e\in E(x)} f(e) \equiv d(x) \equiv \beta(x) \pmod 2$ since $f(e) \equiv 1 \pmod 2$. So if a vertex x does not satisfy Eq. (1), then $f(x) \equiv \beta(x) + 2 \pmod 4$ and we call it a bad vertex.

Let U be the set of all bad vertices. And let $E_i = \{e \in E(G) \mid f(e) = i\}$, $i = 1, 3$.

On one hand

$$\sum_{x\in V(G)} f(x) = 2|E_1| + 6|E_3| \equiv 2|E_1| + 2|E_3| = 2|E(G)| \pmod 4. \tag{9}$$

On the other hand

$$\begin{aligned}\sum_{x\in V(G)} f(x) &\equiv \sum_{x\in U}(\beta(x)+2) + \sum_{x\in V(G)\setminus U} \beta(x) \\ &= 2|U| + \sum_{x\in V(G)} \beta(x) \equiv 2|U| + 2|E(G)| \pmod 4.\end{aligned} \tag{10}$$

By combining Eqs. (9) and (10),

$$2|E(G)| \equiv \sum_{x\in V(G)} f(x) \equiv 2|U| + 2|E(G)| \pmod 4.$$

Hence,

$$|U| \equiv 0 \pmod 2.$$

Let u and v be two vertices of U. Since G is connected, there is a path joining u and v. For each edge e in the path we change $f(e)$ by swapping 1 and 3. It is easy to verify that the number of bad vertices decreases by 2. We repeat the operation until there are no bad vertices. □

Lemma 2.1 is proved as a corollary of Lemma 3.2.

Definition 3.1. The dynamic density of a graph H is the greatest integer k such that

$$\min_{\mathcal{P}} \left\{ \frac{|E(H/\mathcal{P})|}{|V(H/\mathcal{P})|-1} \right\} > k \tag{11}$$

where the minimum is taken over all possible partitions $\mathcal{P}$ of the vertex set of H, and $H/\mathcal{P}$ is the graph obtained from H by shrinking each part of $\mathcal{P}$ into a single vertex.

Lemma 3.2. *Let G be a non-bipartite subgraph. If the dynamic density of G is at least k, then $E(G)$ has a partition $\{T_1, \ldots, T_k, F\}$ such that*

(1) *for each $i = 1, \ldots, k$, the subgraph T_i is a spanning tree, while $F \neq \emptyset$,*

(2) *there is a T_i such that $T_i \cup F$ contains an odd-circuit.*

Lemma 3.2 is a refinement of the following fundamental theorem in graph theory when a graph has some extra edges beyond several spanning trees.

Lemma 3.3 (*Tutte [10] and Nash-Williams [8]*)**.** *A graph H contains k edge-disjoint spanning trees if and only if*

$$\min_{\mathcal{P}} \left\{ \frac{|E(H/\mathcal{P})|}{|V(H/\mathcal{P})|-1} \right\} \geq k \tag{12}$$

where the minimum is taken over all possible partitions $\mathcal{P}$ of the vertex set of H, and $H/\mathcal{P}$ is the graph obtained from H by shrinking each part of $\mathcal{P}$ as a single vertex.

Notice the difference between (11) and (12): the inequality of (11) is *strict*.

By the definition of dynamic density (Inequality (11)), we have the following observation.

Observations. If G is of dynamic density at least k, then for any proper subgraph H of G, the contracted graph G/H remains of dynamic density at least k.

Proof of Lemma 3.2. Let G be a counterexample to the lemma. By Lemma 3.3, let $T_1, \ldots, T_k$ be edge-disjoint spanning trees of G.

Recursively label $E(G)$ as follows.

Rule (1). Starting from all edges $e \in E(G) - (\bigcup_{i=1}^{k} E(T_i))$,

$$\phi(e) = 0;$$

Rule (2). For each $e' \in E(G)$, if $\phi(e') = h$ and $T_i + e'$ contains a circuit $C_{e'}$, then every unlabeled edge e'' of $C_{e'}$ is labeled with $\phi(e'') = h + 1$.

Let H be the maximum subgraph of G consisting of all labeled edges.

Claim 1. *In the contracted graph G/H, each T_i/H is a spanning tree.*

Proof. By the maximality of H and by Rule (2) of the labeling, each T_i/H remains acyclic in G/H.

Claim 2.

$$\bigcup_{i=1}^{k} E(T_i/H) = E(G/H).$$

Proof. The claim follows by Rule (1) of the labeling.

Claim 3.

$$H = G$$

(that is, all edges of G are labeled).

Proof. Assume that H is a proper subgraph of G. Claims 1 and 2 imply that

$$\frac{|E(G/H)|}{|V(G/H)| - 1} = k.$$

This contradicts the observation (that G/H is of dynamic density at least k).

Final step. Let e^* be the edge of G with the smallest label ϕ such that $e^* + T_i$ contains an odd circuit, for some $i \in \{1, \ldots, k\}$. Such edge e^* exists since G is not bipartite. Let $\phi(e^*) = q$.

Note that the integer q is an index for a given partition $\mathcal{X} = \{T_1, \ldots, T_k, F\}$ of G. Denote it by $\omega_{\mathcal{X}}$. *Among all such partitions of G, choose the one $\mathcal{X}$ with the* **smallest** $\omega_{\mathcal{X}}$.

If $\omega_{\mathcal{X}} = 0$, then G is not a counterexample. Hence, assume $\omega_{\mathcal{X}} = q \geq 1$.

Let $\{e_0, \ldots, e_q\}$ be the sequence of edges such that

$$e^* = e_q$$

and, for each $\lambda = q - 1, q - 2, \ldots, 1, 0$,

$$\phi(e_\lambda) = \lambda,$$

and $e_{\lambda+1}$ is an edge contained in the circuit of $T_{i_\lambda} + e_\lambda$, for some $i_\lambda \in \{1, \ldots, k\}$.

Let

$$T'_{i_0} = T_{i_0} + e_0 - e_1, T'_j = T_j \text{ if } j \in \{1, \ldots, k\} - \{i_0\} \text{ and } F' = F - e_0 + e_1.$$

In the new partition $\mathcal{X}' = \{T'_1, \ldots, T'_k, F'\}$, it is easy to see, by Rule (2) of labeling, that the index $\omega_{\mathcal{X}'}$ is reduced since each $\phi(e_i)$ is now reduced by 1 if $i = 2, \ldots, q$. Furthermore, the circuit contained in $e_q + T'_{i_{q-1}}$ remains of odd length. This contradicts the choice of $\mathcal{X}$ that $\omega_{\mathcal{X}}$ is smallest, and therefore, completes the proof of the lemma.

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References

[1] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed, Vertex coloring edge partitions, J. Combin. Theory Ser. B 94 (2005) 237–244.
[2] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, A. Thomason, Vertex-coloring edge-weightings, Combinatorica 27 (2007) 1–12.
[3] L. Addario-Berry, K. Dalal, B.A. Reed, Degree constrained subgraphs, Discrete Appl. Math. 156 (2008) 1168–1174.
[4] J.A. Bondy, U.S.R. Mutry, Graph Theory, Spring, New York, 2008.
[5] M. Kalkowski, M. Karoński, F. Pfender, Vertex-coloring edge-weightings: towards the 1-2-3-conjecture, J. Combin. Theory Ser. B 100 (2010) 347–349.
[6] M. Karoński, T. Luczak, A. Thomason, Edge weights and vertex colors, J. Combin. Theory Ser. B 91 (2004) 151–157.
[7] H. Lu, Q. Yu, C.-Q. Zhang, A note on vertex-coloring 2-edge-weighting, European J. Combin. 32 (1) (2011) 21–27.
[8] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[9] B. Seamone, The 1-2-3 Conjecture and related problems: a Survey, arXiv:1211.5122v1. [math.CO] 21 Nov 2012.
[10] W.T. Tutte, On the problem of decompositing a graph into n connected factors, J. Lond. Math. Soc. 36 (1961) 221–230.
[11] T. Wang, Q.-L. Yu, On vertex-coloring 13-edge-weighting, Front. Math. China 3 (2008) 581–587.
[12] D.B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, NJ, 1996.