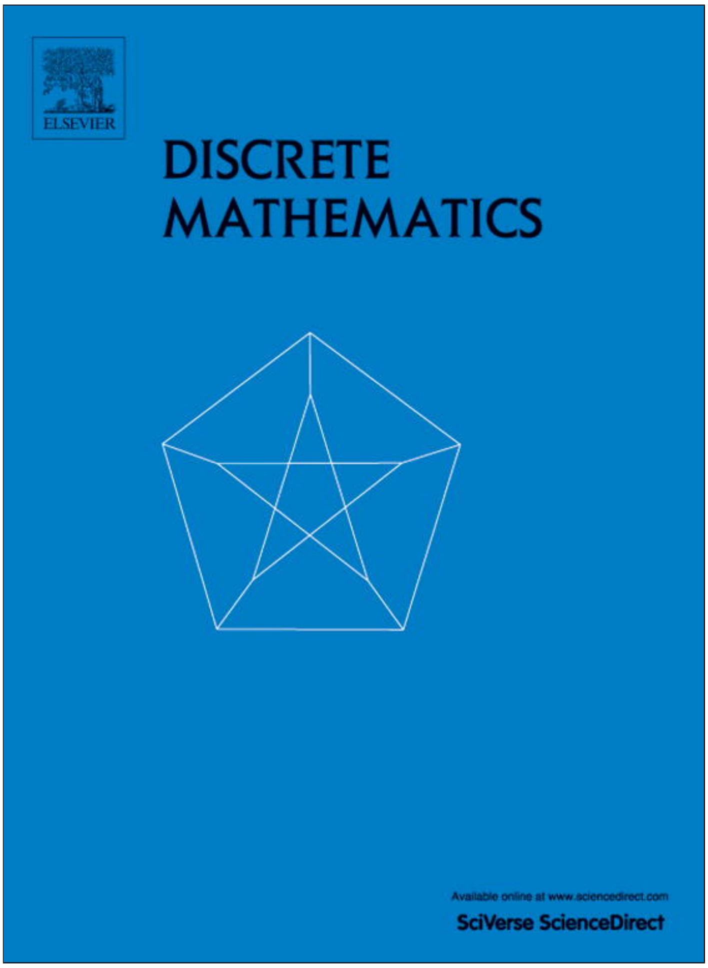
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Circuit extension and circuit double cover of graphs

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ABSTRACT

Let G be a cubic graph and C be a circuit. An extension of C is a circuit D such that $V(C) \subseteq V(D)$ and $E(C) \neq E(D)$. The study of circuit extension is motivated by the circuit double cover conjecture. It is proved by Fleischner (1990) that a circuit C is extendable if C has only one non-trivial Tutte bridge. It is further improved by Chan, Chudnovsky and Seymour (2009) that a circuit is extendable if it has only one odd Tutte bridge. Those earlier results are improved in this paper that C is extendable if all odd Tutte bridges of C are sequentially lined up along C. It was proved that if every circuit is extendable for every bridgeless cubic graph, then the circuit double cover conjecture is true (Kahn, Robertson, Seymour 1987). Although graphs with stable circuits have been discovered by Fleischner (1994) and Kochol (2001), variations of this approach remain one of most promising approaches to the circuit double cover conjecture. Following some early investigation of Seymour and Fleischner, we further study the relation between circuit extension and circuit double cover conjecture, and propose a new approach to the conjecture. This new approach is verified for some graphs with stable circuits constructed by Fleischner and Kochol.

1. Introduction

Let G be a bridgeless graph. A subgraph of G is *even* if every vertex is of even degree. A *circuit* of G is a connected 2-regular graph. The following is the well-known Circuit Double Cover Conjecture.

Conjecture 1.1 ([19,23,26,29]). *Every bridgeless graph G has a family of circuits that covers every edge precisely twice.*

Circuit Double Conjecture has been verified for K_5-minor-free graphs, Petersen-minor-free graphs [1,2] and graphs with specific structures such as Hamiltonian path [27], small oddness [18,16,17] and spanning subgraphs [13–15,30]. It suffices to show that the Circuit Double Cover Conjecture holds for cubic graphs [20]. The Circuit Double Cover Conjecture is strengthened to the Strong Circuit Double Cover Conjecture as follows.

Conjecture 1.2 (Seymour, See [7, p. 237], [8], Also See [13]). *Let G be a bridgeless cubic graph and C be a circuit of G. Then G has a circuit double cover which contains C.*

The Strong Circuit Double Cover Conjecture is related to Sabidussi's Compatibility Conjecture which asserts that if T is a Eulerian trail of a Eulerian graph G of minimum degree at least 4, there exists a circuit decomposition $\mathscr{D}$ of G such that no transition of T is contained in any element of $\mathscr{D}$. A circuit C of a graph G is a *dominating circuit* if $G - V(C)$ has no edges. Sabidussi's Compatibility Conjecture is equivalent to the following circuit cover version.

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Conjecture 1.3 (*Sabidussi and Fleischner [9], and Conjecture 2.4 in [3, p. 462]*). *Let G be a bridgeless cubic graph and C be a dominating circuit of G. Then G has a circuit double cover which contains C.*

There are few partial results known for Conjectures 1.2 and 1.3. It is well-known that 3-edge-colorable cubic graphs satisfy Conjectures 1.2 and 1.3. The following result is obtained by Fleischner.

Theorem 1.4 (*Fleischner [10], Also See [12]*). *Let G be a cubic graph with a circuit C such that $G - V(C)$ has only one vertex. Then G has a circuit double cover containing C.*

One way to attack these conjectures is circuit extension. This idea was first proposed by Seymour (see [11,22]). Given a circuit C of a bridgeless cubic graph G, a circuit D is called an *extension* of C if $V(C) \subseteq V(D)$ and $E(D) \neq E(C)$. If C has an extension in G, the pair (G, C) is *extendable*. The following is a problem proposed by Seymour.

Problem 1.5. Let G be a bridgeless cubic graph and C be a circuit. Is (G, C) extendable?

If the answer to Problem 1.5 is yes, then Conjectures 1.1–1.3 will follow (Proposition 6.1.4 in [31, p. 67]). However, Fleischner [11] constructed a counterexample to Problem 1.5 and answered Seymour's problem negatively. After that, Kochol [22] constructed an infinite family of cyclic 4-edge-connected cubic graphs G with circuits C such that (G, C) is not extendable. But it is still interesting to ask which cubic graphs have the circuit extension property.

Let G be a cubic graph and C be a circuit of G. Each component B of $G - E(C)$ is called a *Tutte-bridge* of C. A vertex in $B \cap C$ is called an *attachment* of B on C. A chord of C is a trivial Tutte-bridge. The *order* of B is the number of vertices in $B - V(C)$. A Tutte-bridge is *odd* if its order is odd.

Theorem 1.6 (*Fleischner [10]*). *Let G be a bridgeless cubic graph and C be a circuit of G. Then the circuit C is extendable if C has only one non-trivial Tutte-bridge.*

Theorem 1.7 (*Chan, Chudnovsky and Seymour [4]*). *Let G be a bridgeless cubic graph and C be a circuit of G. Then the circuit C is extendable if C has only one odd Tutte-bridge.*

In this paper, the above results are further strengthened in Theorem 1.9.

Definition 1.8. Let G be a bridgeless cubic graph and C be a circuit of G. All odd Tutte-bridges of C are *sequentially lined up along* C if each odd Tutte-bridge Q_i $(1 \le i \le t)$ of C has an attachment v_i such that $v_i v_{i+1} \in E(C)$ for $1 \le i \le t-1$.

Theorem 1.9. *Let G be a bridgeless cubic graph and C be a circuit of G. The circuit C is extendable if all odd Tutte-bridges of C are sequentially lined up along C.*

2. Proof of the main theorem

Let G be a cubic graph and M be a subset of $E(G)$. Let $G - M$ be the subgraph of G obtained from G by deleting all edges in M. The *suppressed graph* $\overline{G - M}$ is a graph obtained from $G - M$ by suppressing all vertices of degree two. If M is a matching, $\overline{G - M}$ is a cubic graph. If M has only one edge e, we use $G - e$ and $\overline{G - e}$ instead.

The following theorem will be used in the proof of the main theorem.

Theorem 2.1 (*Smith's Theorem, [28]*). *Let G be a cubic graph. Then every edge of G is contained in an even number of Hamiltonian circuits.*

Now, we are ready to prove the main theorem.

Proof of Theorem 1.9. Suppose that (G, C) is a minimum counterexample with $|E(G)|$ as small as possible. Let $T_1, \ldots, T_k$ be all odd Tutte-bridges of C such that each T_i has an attachment v_i and $v_i v_{i+1} \in E(C)$ for $i = 1, \ldots, k-1$.

(1) *For any edge $e \in E(G) \setminus E(C)$, $G - e$ is bridgeless.*

If not, assume that $e = uv \in E(G) \setminus E(C)$ is such that $G - e$ has a bridge $e' = u'v'$. Clearly, e is not a chord of C and neither is e'. Then $G - \{e, e'\}$ has two components Q and Q'. Without loss of generality, assume that $C \subseteq Q$ and $u, u' \in Q$. By parity, Q' is of even order. Let G' be the new cubic graph obtained from Q by adding a new edge uu'. Note that $T_1 \cap G', \ldots, T_k \cap G'$ are all odd Tutte-bridges of C in G', and $v_i v_{i+1}$ is still an edge of C in G'. Since $|E(G')| < |E(G)|$, (G', C) is extendable. Let D be an extension of C in G'. If D does not contain uu', then D is also an extension of C in G, a contradiction. If D contains uu', then $D - uu' + \{e, e'\} + P_{vv'}$ is an extension of C in G, where $P_{vv'}$ is a path of Q' joining v and v', a contradiction. The contradiction implies that $G - e$ is bridgeless.

(2) *Every non-trivial Tutte-bridge Q is acyclic.*

Suppose to the contrary that C has a Tutte-bridge Q which has a circuit. Let $e = uv$ be an edge on the circuit. Then the order of $\overline{Q - e}$ has the same parity as the order of Q. By (1), $\overline{G - e}$ is bridgeless. Note that the odd Tutte-bridges of C in $\overline{G - e}$ have the same property as the odd Tutte-bridges of C in G. Since $|E(\overline{G - e})| < |E(G)|$, C has an extension D in $\overline{G - e}$. Let D' be the corresponding circuit of D in G, which is an extension of C in G, a contradiction.

(3) *The circuit C is dominating.*

By (2), every non-trivial Tutte-bridge is a tree. We have to show that every non-trivial Tutte-bridge Q is $K_{1,3}$. Choose a leaf v of $Q - V(C)$ such that $v \notin N(v_i)$ for all $1 \le i \le t$, and let $e = uv$ be an edge of $Q - V(C)$. Then $Q - e$ has two

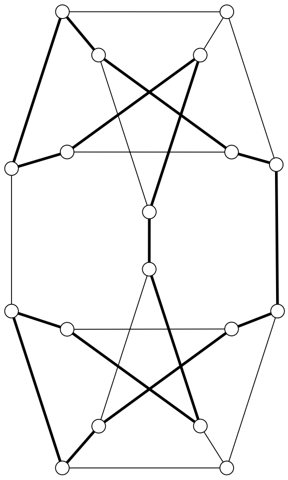

Fig. 1. Fleischner's counterexample and a circuit without extension.

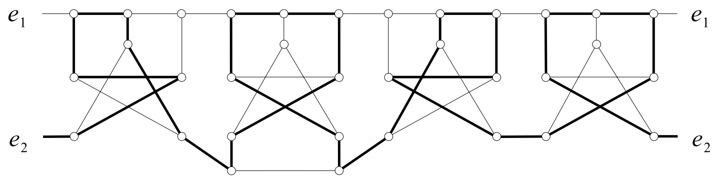

Fig. 2. Kochol's counterexample and a circuit without extension.

components, Q_1 and Q_2: (i) $u, v_i \in V(\overline{Q_1})$ and $|V(\overline{Q_1})| \equiv |V(Q)| (\text{mod } 2)$; (ii) $v \in V(\overline{Q_2})$ and $\overline{Q_2}$ is a chord (joining two vertices of $N(v) \cap V(C)$). By (1), $\overline{G-e}$ is bridgeless and the odd Tutte-bridges of C in $\overline{G-e}$ have the same property as the odd Tutte-bridges of C in G. Note that $|E(\overline{G-e})| < |E(G)|$. Hence C has an extension D in $\overline{G-e}$. So the corresponding circuit D' of D in G is an extension of C in G, a contradiction. The contradiction implies that $Q - V(C)$ has no edges. Since G is cubic, $Q - V(C)$ is an isolated vertex. Hence Q is isomorphic to $K_{1,3}$.

(4) *The final step.*

By (3), G has only odd Tutte-bridges $T_1, \ldots, T_k$ and each T_i has only one vertex x_i which is not on C. Note that v_i is an attachment of T_i. So $x_i v_i \in E(T_i)$. Let $M := \{x_i v_i | i = 1, \ldots, t\}$. Then the suppressed circuit C' of C is a Hamiltonian circuit of $\overline{G-M}$, and the path $v_1^- v_1 v_2 \cdots v_t v_t^+$ of C is corresponding to an edge $v_1^- v_t^+$ in $\overline{G-M}$. By Smith's Theorem (Theorem 2.1), the edge $v_1^- v_t^+$ is contained in another Hamilton circuit D' which is an extension of C' in $\overline{G-M}$. Let D be the circuit of G which is corresponding to D'. Then D is an extension of C in G, a contradiction. This completes the proof. □

3. Circuit extension and circuit double cover

Circuit extension of graphs is an approach to solve the Circuit Double Cover Conjecture. Instead of cubic graphs, we pay more attention to bridgeless subgraphs of cubic graphs which are called *subcubic graphs*. Let H be a subcubic graph and V_3 be the set of all degree 3 vertices in H. An *extension* of C is a circuit D such that $V_3(C) \subseteq V_3(D)$ and $E(C) \neq E(D)$, where $V_3(C) = V(C) \cap V_3$ and $V_3(D) = V(D) \cap V_3$. Note that, if D is an extension of a circuit C of a subcubic graph G, then $H - (E(C) - E(D))$ is again subcubic.

Proposition 3.1 (*Kahn, Robertson and Seymour [21], Also see [4,24,25]*). *If Problem 1.5 is true for every circuit of all 2-connected cubic graphs, then the Circuit Double Cover Conjecture is true.*

Proposition 3.1 can be proved by recursively applying Problem 1.5 as follows.

- Let $G_0 := G$ and C_0 be a circuit of G;
- Find a C_0-extension C_1 in G and let $G_1 := G_0 - (E(C_0) - E(C_1))$;
- ···
- Find a C_{i-1}-extension C_i in G_{i-1} and let $G_i := G_{i-1} - (E(C_{i-1}) - E(C_i))$;
- ···
- Until $G_{t-1} - (E(C_{t-2}) - E(C_{t-1}))$ is a circuit, denoted by C_t.

This process is called a *circuit-extension process* with the output

$$\{C_0, \ldots, C_t\}$$

which is called *an extension sequence of* C_0.

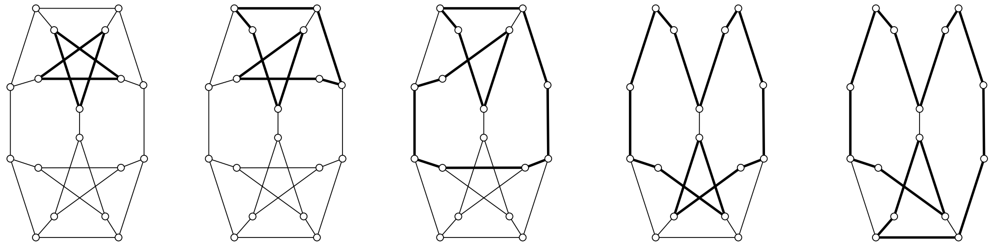

Fig. 3. Fleischner's counterexample and a circuit-extension process.

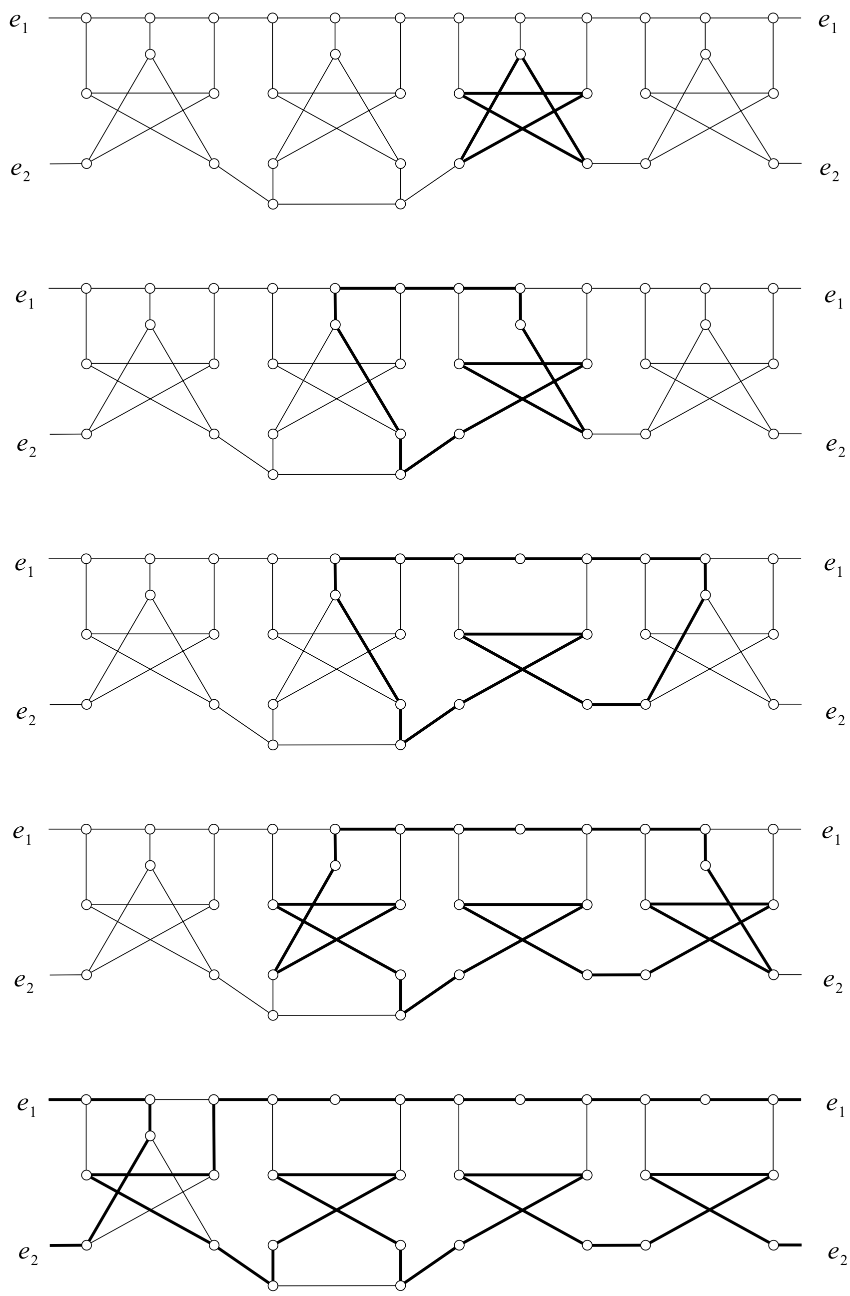

Fig. 4. Kochol's counterexample and a circuit-extension process.

Proposition 3.1 can be restated as follows.

Proposition 3.2. *Let G be a bridgeless cubic graph and C_0 be a given circuit of G. Assume that C_0 has an extension sequence $\{C_0, C_1, \ldots, C_t\}$. If, for every bridgeless subgraph H of G, Problem 1.5 is true for the suppressed cubic graph $\overline{H}$, then G has a circuit double cover*

$$\{C_0, C_0 \triangle C_1, \ldots, C_{t-1} \triangle C_t, C_t\}.$$

However, Problem 1.5 is not true in general: counterexamples (G, C_0) exist (see [11,22], also Figs. 1 and 2). That is, for some graphs G, there is a circuit C_0 which does not have an extension sequence. In order to avoid such non-extendable circuit (stable circuit), variations of circuit extensions have been studied in [5,6].

Without specifying the initial circuit C_0, we suggest a modified approach as follows (another variation is proposed in the final section).

Problem 3.3. For a given graph G, is there a circuit C_0 which has an extension sequence?

With the same argument as Proposition 3.2, Problem 3.3 implies the Circuit Double Cover Conjecture. For Problem 1.5, Fleischner and Kochol discovered some counterexamples. We further verify that they are not counterexamples for Problem 3.3 in the following two propositions.

Proposition 3.4. *Fleischner's counterexample (see Fig. 1) has a circuit C_0 such that the circuit-extension process can be carried out.*

Proposition 3.5. *Kochol's counterexample (see Fig. 4) has a circuit C_0 such that the circuit-extension process can be carried out.*

Proofs of Propositions 3.4 and 3.5 are illustrated in Figs. 3 and 4, respectively. The circuits in the extension sequences are in bold lines. The circuit in the last step is a Hamilton circuit C of a subcubic graph whose suppressed graph is a cubic graph with a Hamilton circuit C. By Smith's Theorem, every cubic graph with a Hamilton circuit C has another Hamilton circuit which is an extension of C. Hence the circuit-extension process can be carried out.

4. Remarks

Because of the existence of counterexamples to Problem 1.5, the circuit-extension process may not be carried out to the end: the process stops when C_{i-1} has no extension in G_{i-1}. We further propose a modification of the circuit-extension process with the following additional requirements.

(1) The initial circuit C_0 is not given, and is a shortest circuit in G.

(2) Among all extensions of C_{i-1} in G_{i-1}, we choose a shortest one to be C_i.

And we conjecture that the modified circuit-extension process can be carried out to the end.

We notice that all those cubic graphs with non-extendable circuits [11,22] contain the Petersen graph as a minor. We propose another related problem: Problem 1.5 *is true for all Petersen-minor free graphs.*

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References

[1] B. Alspach, Cycle covers of cubic multigraphs, Discrete Math. 111 (1993) 11–17.
[2] B. Alspach, L. Goddyn, C.-Q. Zhang, Graphs with the circuit cover property, Tran. Amer. Math. Soc. 344 (1994) 131–154.
[3] B. Alspach, C. Godsil, Unsolved problems, in: Cycles in Graphs, in: Ann. Discrete Math., vol. 27, North-Holland, Amsterdam, 1985, pp. 461–467.
[4] M. Chan, A survey of the cycle double cover conjecture, Princeton University, 2009, Preprint.
[5] E.G.M. Esteva, T.R. Jensen, On semiextensions and circuit double covers, J. Combin. Theory Ser. B 97 (2007) 474–482.
[6] E.G.M. Esteva, T.R. Jensen, A note on semiextensions of stable circuits, Discrete Math. 309 (2009) 4952–4954.
[7] H. Fleischner, Cycle decompositions, 2-coverings, removable cycles and the four-color disease, in: J.A. Bondy, U.S.R. Murty (Eds.), Progress in Graph Theory, Academic Press, New York, 1984, pp. 233–246.
[8] H. Fleischner, Proof of the strong 2-cover conjecture for planar graphs, J. Combin. Theory Ser. B 40 (1986) 229–230.
[9] H. Fleischner, Eulerian graph, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory (2), Academic Press, London, 1983, pp. 17–53.
[10] H. Fleischner, Communication at Cycle Double Cover Conjecture Workshop, Barbados, February 25–March 4, 1990.
[11] H. Fleischner, Uniqueness of maximal dominating cycles in 3-regular graphs and Hamiltonian cycles in 4-regular graphs, J. Graph Theory 18 (1994) 449–459.
[12] H. Fleischner, R. Häggkvist, Circuit double covers in special types of cubic graphs, Discrete Math. 309 (2009) 5724–5728.
[13] L. Goddyn, Cycle covers of graphs, Ph.D. Thesis, University of Waterloo, 1988.
[14] R. Häggkvist, K. Markström, Cycle double covers and spanning minors I, J. Combin. Theory Ser. B 96 (2006) 183–206.
[15] R. Häggkvist, K. Markström, Cycle double covers and spanning minors II, Discrete Math. 306 (2006) 762–778.
[16] R. Häggkvist, S. McGuinness, Double covers of cubic graphs with oddness 4, J. Combin. Theory Ser. B 93 (2005) 251–277.
[17] A. Huck, On cycle-double covers of graphs of small oddness, Discrete Math. 229 (2001) 125–165.
[18] A. Huck, M. Kochol, Five cycle double covers of some cubic graphs, J. Combin. Theory Ser. B 64 (1995) 119–125.
[19] A. Itai, M. Rodeh, Covering a graph by circuits, in: Automata, Languages and Programming, in: Lecture Notes in Computer Science, vol. 62, Springer-Verlag, Berlin, 1978, pp. 289–299.
[20] F. Jaeger, A survey of the cycle double cover conjecture, in: B. Alspach, C. Godsil (Eds.), Cycles in Graphs, in: Ann. Discrete Math., vol. 27, North-Holland, Amsterdam, 1985, pp. 1–12.
[21] J. Kahn, N. Robertson, P.D. Seymour, et al. Communication at Bellcore, 1987.
[22] M. Kochol, Stable dominating circuits in snarks, Discrete Math. 233 (2001) 247–256.
[23] P.D. Seymour, Sums of circuits, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, pp. 342–355.
[24] P.D. Seymour, Communication at Cycle Double Cover Conjecture Workshop, Barbados, February 25–March 4, 1990.
[25] P.D. Seymour, Personal communication, 2012.

[26] G. Szekeres, Polyhedral decompositions of cubic graphs, Bull. Austral. Math. Soc. 8 (1973) 367–387.
[27] M. Tarsi, Semi-duality and the cycle double cover conjecture, J. Combin. Theory Ser. B 41 (1986) 332–340.
[28] A. Thomason, Hamiltonian Cycles and uniquely edge colorable graphs, Ann. Discrete Math. 3 (1978) 259–268.
[29] W.T. Tutte, Personal correspondence to H. Fleischner, July 22, 1987.
[30] D. Ye, C.-Q. Zhang, Cycle double covers and the semi-kotzig frame, Europ. J. Combin. 33 (4) (2012) 624–631.
[31] C.-Q. Zhang, Circuit Double Cover of Graphs, Cambridge University Press, Cambridge, 2012.