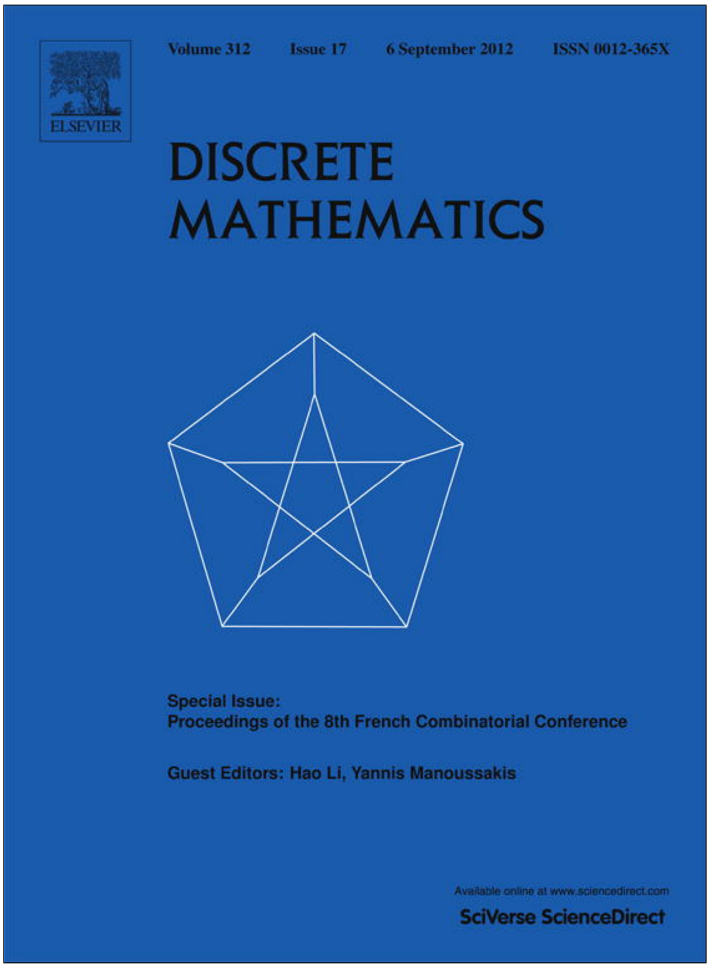
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The edge spectrum of the saturation number for small paths[☆]

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ABSTRACT

Let H be a simple graph. A graph G is called an H-saturated graph if H is not a subgraph of G, but adding any missing edge to G will produce a copy of H. Denote by $SAT(n, H)$ the set of all H-saturated graphs G with order n. Then the saturation number $sat(n, H)$ is defined as $\min_{G\in SAT(n,H)} |E(G)|$, and the extremal number $ex(n, H)$ is defined as $\max_{G\in SAT(n,H)} |E(G)|$. A natural question is that of whether we can find an H-saturated graph with m edges for any $sat(n, H) \le m \le ex(n, H)$. The set of all possible values m is called the edge spectrum for H-saturated graphs. In this paper we investigate the edge spectrum for P_i-saturated graphs, where $2 \le i \le 6$. It is trivial for the case of P_2 that the saturated graph must be an empty graph.

1. Introduction and notation

In this paper we only consider graphs without loops or multiple edges. For terms not defined here see [3]. For a graph G we use G to represent the vertex set $V(G)$ and the edge set $E(G)$ when the meaning is clear from the context. Furthermore, $|V(G)| = n$, unless otherwise specified. Also, K_p denotes the complete graph on p vertices. A graph G is termed an (n, m) graph if $|V(G)| = n$ and $|E(G)| = m$.

A fixed graph G is called an H-*saturated* graph if the graph H is not a subgraph of G, but adding any missing edge to G will produce a copy of H. The collection of all H-saturated graphs of order n is denoted by $SAT(n, H)$, and the *saturation number*, denoted as $sat(n, H)$, is the minimum number of edges of a graph in the set $SAT(n, H)$. The graphs in $SAT(n, H)$ with the minimum number of edges will be denoted by $\underline{SAT}(n, H)$. The saturation number was introduced by Erdős, Hajnal, and Moon in [4] in which the authors proved $sat(n, K_p) = \binom{p-2}{2} + (n-p+2)(p-2)$ and $\underline{SAT}(n, K_p) = \{K_{p-2} \vee \overline{K}_{n-p+2}\}$, where $\vee$ is the standard graph joining operation. The maximum number of edges of a graph from $SAT(n, H)$ is the well known Turán extremal number (see [9]), and is usually denoted by $ex(n, H)$. The parameters $sat(n, H)$ and $ex(n, H)$ have been investigated for a range of graphs H. Generalizations to hypergraphs also exist (see [7]).

A natural aim is to find, if possible, an H-saturated graph with m edges for any integer m between the saturation number and extremal number. Barefoot et al. [2] studied the edge spectrum of K_3-saturated graphs and proved the following result.

Theorem 1.1 ([2]). *Let $n \ge 5$ and m be nonnegative integers. There is an (n, m) K_3-saturated graph if and only if $2n-5 \le m \le \lfloor (n-1)^2/4 \rfloor + 1$ or $m = k(n-k)$ for some positive integer k.*

Theorem 1.1 says that a K_3-saturated graph is either a complete bipartite graph or its size falls in the given range and all values in this range are possible.

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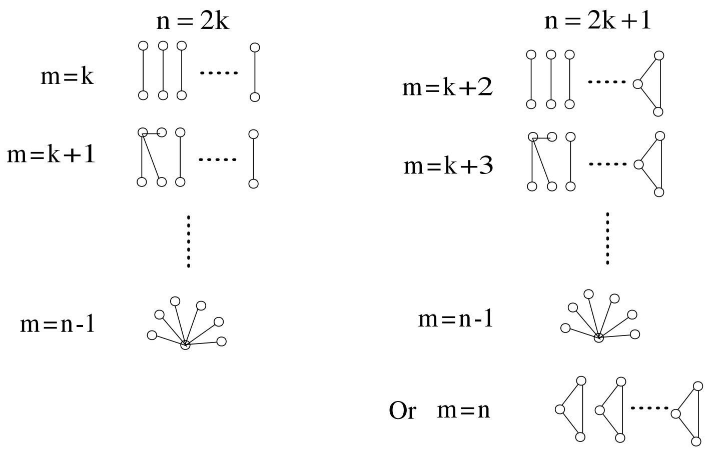


Fig. 1. Description of P_4-saturated graphs.

Later Amin [1] extended this result from K_3-saturated graphs to any complete graph K_p in her Ph.D. thesis, which is the starting point of this paper. In Section 2, we present the known results on saturation numbers and extremal numbers for paths, and then investigate the properties of connected P_5-saturated graphs and connected P_6-saturated graphs. In Sections 3 and 4 we shall give a complete characterization of the edge spectrum for P_5-saturated graphs and P_6-saturated graphs respectively.

2. Known results

In [6] Kászonyi and Tuza proved several general results concerning saturated graphs including an upper bound for $sat(n, H)$ for any connected graph H by constructing an H-saturated graph.

Theorem 2.1 ([6]). *Saturation numbers for paths:*

(a) *For* $n \geq 3$, $sat(n, P_3) = \lfloor n/2 \rfloor$.
(b) *For* $n \geq 4$, $sat(n, P_4) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$
(c) *For* $n \geq 5$, $sat(n, P_5) = \lceil \frac{5n-4}{6} \rceil$.
(d) *Let* $a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if } k = 2t, \\ 4 \cdot 2^{t-1} - 2 & \text{if } k = 2t+1. \end{cases}$ *If* $n \geq a_k$ *and* $k \geq 6$, *then* $sat(n, P_k) = n - \lfloor \frac{n}{a_k} \rfloor$.

In this paper, we will mainly consider the cases of P_5 and P_6. By Theorem 2.1 we have $sat(n, P_6) = \lceil 9n/10 \rceil$ for any $n \geq 10$.

Next let us recall a result concerning the Turán extremal number, which was proved by Faudree and Schelp [5] in 1975.

Theorem 2.2 ([5]). *If G is a graph with $|V(G)| = kt + r$, $0 \leq r < k$, containing no path on $k+1$ vertices, then $|E(G)| \leq t\binom{k}{2} + \binom{r}{2}$ with equality if and only if G is either $(tK_k) \cup K_r$ or $((t-l-1)K_k) \cup (K_{(k-1)/2} \vee \overline{K}_{(k+1)/2+lk+r})$ for some l, $0 \leq l < t$, where k is odd, $t > 0$, and $r = (k \pm 1)/2$.*

Corollary 2.3 ([8]). *For all integer n, $n \geq 3$,*

(a) $ex(n, P_4) = \begin{cases} n & n \equiv 0 \pmod 3 \\ n-1 & n \equiv 1, 2 \pmod 3 \end{cases}$
(b) $ex(n, P_5) = \begin{cases} 3n/2 & n \equiv 0 \pmod 4 \\ 3n/2 - 2 & n \equiv 2 \pmod 4 \\ 3(n-1)/2 & n \equiv 1, 3 \pmod 4 \end{cases}$
(c) $ex(n, P_6) = \begin{cases} 2n & n \equiv 0 \pmod 5 \\ 2n-2 & n \equiv 1, 4 \pmod 5 \\ 2n-3 & n \equiv 2, 3 \pmod 5. \end{cases}$

Considering the fact that for any P_3-saturated graph G, no two edges can be incident to each other and G contains at most one isolated vertex, therefore, $sat(n, P_3) = ex(n, P_3) = \lfloor n/2 \rfloor$. As for the case of P_4-saturated graphs, the following figures clearly show how we can evolve a P_4-saturated graph with the least edges to one with the most edges.

From the Fig. 1 we can evolve a P_4-saturated graph, a perfect matching or a matching union a triangle to one with $n-1$ edges for any integer n. In addition, when $n = 3p$ we can take $G = pK_3$ and find one more P_4-saturated graph of size n.

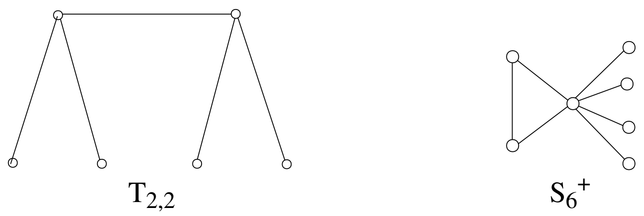


Fig. 2. Two possible structures for a connected P_5-saturated graph.

3. Edge spectra of P_5-saturated graphs

If G is a P_5-saturated graph with order less than 5, then G must be a complete graph. Furthermore, the order of the union of any two components of G must be at least 5 since otherwise we can add an edge joining those two components and have no copies of P_5 in the resulting graphs. Therefore the structure of every component of a P_5-saturated graph becomes important and we have the following lemma concerning it.

Lemma 3.1. *If H is a connected P_5-saturated graph with order at least 5, then each block of H is a clique of order 2 or 3.*

Proof. If H is 2-connected, i.e. H consists of only one block, H cannot be a complete graph since H is P_5-saturated graph. Hence we may take two vertices u and v with $uv \notin E(H)$. There is no circuit with length more than 4 since otherwise a subgraph P_5 will be found. Therefore, we can find a circuit of length 4. Considering that $n \geq 5$, there is a vertex w connecting to some vertex on this circuit and forming a path P_5. Thus H cannot be 2-connected.

Let B be a block of H, then the order of B must be less than 4 since otherwise we may find a circuit of length at least 4 within block B, which forms a P_5 with any one edge outside B. Thus the order of each block in H is either 3 or 2, as we desired. □

Let H be a connected P_5-saturated graph. It is trivial for the case $|V(H)| \leq 4$ that H must be a complete graph. Hence we may assume that $|V(H)| \geq 5$. If H contains two triangles T_1 and T_2, which are connected by a path Q, then a path of order at least 5 is naturally contained as a subgraph. Hence H contains at most one triangle. Therefore, by Lemma 3.1 either G is a tree or a graph obtained by adding exactly one edge to a star (see Fig. 2). Then we get the following lemmas.

Lemma 3.2. *If H is a component of a P_5-saturated graph other than K_4, then the size of H is either $n'-1$ or n', where $n' = |V(H)|$.*

Let G be a P_5-saturated graph and S_μ^+ be a graph obtained from the star S_μ by adding one edge. If we denote by α_1 the number of acyclic components of G, by α_2 the number of S_μ^+'s, and by α_3 the number of $K_4's$ in G, then by applying the above lemma we have the next lemma immediately.

Lemma 3.3. *If $G \in SAT(n, P_5)$, then $|E(G)| = n - \alpha_1 + 2\alpha_3$, where α_1, α_3 are defined above.*

The key idea of our method is constructing a new P_5-saturated graph from an existing smaller P_5-saturated graph. Our main result on P_5 is heavily dependent on the following lemmas.

Lemma 3.4. *Let $m \geq n \geq 5$. There is an (n, m) graph G in $SAT(n, P_5)$ if and only if there exists an $(n+4, m+6)$ graph G' in $SAT(n+4, P_5)$.*

Proof. It is trivial that $G \in SAT(n, P_5)$ implies $G' = G \cup K_4 \in SAT(n+4, P_5)$, where $\cup$ is the graph union operation and $V(G') = V(G) \cup V(K_4)$, $E(G') = E(G) \cup E(K_4)$. For the necessity, we assume that G'' is an $(n+4, m+6)$ graph in $SAT(n+4, P_5)$. If there exists a component K_4 in G'', then $G = G'' - K_4$ will be an (n, m) graph in $SAT(n, P_5)$. Hence, we may suppose that G'' contains no K_4's, i.e. $\alpha_3 = 0$. By Lemma 3.3, $m + 6 = |E(G'')| = |V(G'')| - \alpha_1 \leq |V(G'')| = n + 4$, contradicting that $m \geq n$. □

So far we have figured out P_5-saturated graphs with $m \geq n$. The next lemma will tell us more information about P_5-saturated graphs with fewer edges.

Lemma 3.5. *Let n be an integer that is at least 5 and $\lceil \frac{5n-4}{6} \rceil \leq m \leq n-1$. There exists a P_5-saturated (n, m) graph.*

Proof. If $n = 5$, then $\lceil \frac{5n-4}{6} \rceil = n - 1 = 4$ and we take $G = K_2 \cup K_3 \in SAT(n, P_5)$. Next we assume that $n > 5$ and write $n = 6k + i$, where $0 \leq i \leq 5$.

First we construct a P_5-saturated graph of size $m = \lceil \frac{5n-4}{6} \rceil$. Let $T_{a,b}$ be a graph obtained from two stars S_a and S_b by adding an edge joining the two centers (see Fig. 2). We can construct the following graphs: $kT_{2,2}$, $(k-1)T_{2,2} \cup T_{2,3}$, $kT_{2,2} \cup K_2$, $(k-1)T_{2,2} \cup T_{2,3} \cup K_2$, $(k-1)T_{2,2} \cup T_{3,3} \cup K_2$, $(k-1)T_{2,2} \cup T_{4,3} \cup K_2$ in line with the i-values. For example, we can take $G = T_{2,2} \cup T_{4,3} \cup K_2$ if $n = 17$.

Then for the values of $m \in (\lceil \frac{5n-4}{6} \rceil, n)$, we can build a P_5-saturated (n, m) graph by the following process starting with saturated graphs with $\lceil \frac{5n-4}{6} \rceil$ edges: $T_{a,b} + T_{x,y} \Rightarrow T_{x+a+1,y+b+1}$ or $T_{x,y} + K_2 \Rightarrow T_{x+1,y+1}$. This process is shown in Fig. 3. At the end of this process we shall get a P_5-saturated with $n - 1$ edges, namely $T_{a,n-a-2}$. □

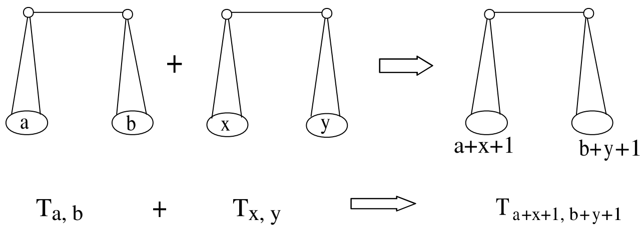


Fig. 3. The evolution of P_5-saturated trees.

Now we are ready to give the edge spectra of P_5-saturated graphs and present one of the main theorems. The interval $[A, B] = [sat(n, P_5), ex(n, P_5)]$ can be obtained from Theorem 2.1 and Corollary 2.3. We want to determine whether this interval is the spectrum of P_5—or are there any missing values within this interval?

It is worth noting here that at each induction process stage we jump four steps. First we give the results for four initial values of n, which are listed in the table, and then apply the induction process to get results for the next four n values.

Theorem 3.1. *Let $n \geq 5$ and $sat(n, P_5) \leq m \leq ex(n, P_5)$ be integers. There exists an (n, m) graph $G \in SAT(n, P_5)$ if and only if $n = 1, 2 \pmod 4$, or*

$$m \notin \begin{cases} \left\{ \dfrac{3n-5}{2} \right\} & \text{if } n \equiv 3 \pmod 4 \\ \left\{ \dfrac{3n}{2} - 3, \dfrac{3n}{2} - 2, \dfrac{3n}{2} - 1 \right\} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Proof. The proof of this result follows from Lemma 3.1 through Lemma 3.5 and the next table, by induction on n.

n	$[A, B]$	$SAT(n, P_5)$	n	$[A, B]$	$SAT(n, P_5)$
5	$[4, 6]$	4: $K_3 \cup K_2$	6	$[5, 7]$	5: $T_{2,2}$
		5: S_4^+			6: $2K_3$
		6: $K_4 \cup K_1$			7: $K_4 \cup K_2$
7	$[6, 9]$	6: $T_{2,3}$	8	$[6, 12]$	6: $T_{2,2} \cup K_2$
		7: $2K_3$			7: $T_{3,3}$
		8: $\emptyset$			8: $K_3 \cup S_4^+$
		9: $K_4 \cup K_3$			9–11: $\emptyset$
					12: $2K_4$

In the above table, the symbol $'\emptyset'$ stands for no $(n, m)P_5$-saturated graph existing and $A = sat(n, P_5)$, $B = ex(n, P_5)$. The nonexistence of P_5-saturated graphs comes from Lemma 3.3 by counting the edges.

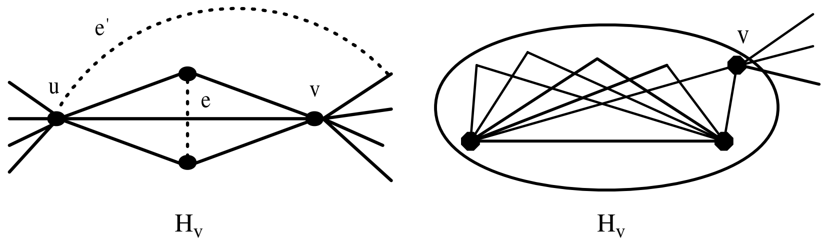
The initial results for $n = 5, 6, 7, 8$ are listed in the above table. By the induction hypothesis, suppose that we have the result for n; then we shall apply Lemma 3.4 to get a result for $n + 4$. Once we have determined all possible P_5-saturated graphs with n vertices, we can also determine the P_5-saturated graphs with $n+4$ vertices according to Lemma 3.4. Therefore we can cover the interval $[sat(n, P_5) + 6, ex(n, P_5) + 6]$ which is exactly the interval $[sat(n, P_5) + 6, ex(n + 4, P_5)]$. In order to finish the argument that we can determine all integers between $sat(n+4, P_5)$ and $ex(n+4, P_5)$, we also need to deal with the subinterval $[sat(n+4, P_5), sat(n, P_5) + 5]$, which is fortunately covered by Lemma 3.5 since $sat(n, P_5) + 5 \leq (n+4) - 1$ for any integer $n \geq 9$. $\square$

4. Edge spectra of P_6-saturated graphs

If G is a P_6-saturated graph with order less than 6, then G must be a complete graph. Furthermore, the order of the union of any two components of G must be at least 6. Let B_n denote the *book* graph, the union of n triangles sharing one edge. A θ-graph is the union of three internally disjoint (simple) paths that have the same two distinct end vertices. As we did for the case of P_5-saturated graph, we shall pay attention to each connected component of G.

Lemma 4.1. *Let H be a (2)-connected P_6-saturated graph of order at least (6). Then H must be a book.*

Proof. Let C be the longest circuit in H. The length of C must be less than 5 since otherwise P_6 would be found in a subgraph obtained from C and any other edge touching it. On the other hand, the length of C cannot be less than 4 since H is 2-connected and every vertex outside of the circuit C is connected to C by two edge-disjoint paths. Therefore C is a circuit of length 4 and we define $C = uxvy$ with $e = uv \notin E(H)$ the missing edge in H, since otherwise this K_4 plus any other vertex would contain a larger circuit in H.



(a) Case 1: $|H_v| = 4$. (b) Case 2: $|H_v| \geq 5$.

Fig. 4. The local structure of H_v.

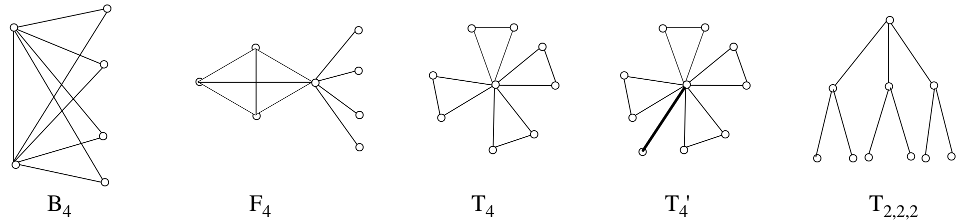


Fig. 5. All possible structures for connected P_6-saturated graphs.

Considering the fact that every edge in H must have an endpoint in C, then the vertex set $H_C = H - C$ is an independent set. Take $w \in H_C$; then w is adjacent to at least two vertices in C from the fact that H is 2-connected, and w cannot be connected to two consecutive vertices in C since otherwise we can find a circuit larger than C. Thus w is connected to either u and v or x and y. Since we need to avoid the appearance of P_6, all vertices from H_C must be connected to either u and v or x and y.

C cannot be an induced 4-circuit since no matter how we connect H_C to C, we can add a new edge—either e or f—back to H. Thus $f = xy \in E(H)$, and all vertices in H_C must connect to x and y. Therefore the resulting graph is a book graph with common edge $f = xy$. $\square$

Lemma 4.2. *If H is a connected P_6-saturated graph with cut vertices, then each block of H must be a clique. Furthermore, if H_1 and H_2 are two blocks in H, then $|V(H_1)| + |V(H_2)| \leq 6$.*

Proof. The result is trivial if H is a tree. Let v be the cut vertex and H_v be one of nontrivial blocks containing v in H. If $|H_v| = 3$, then H_v is complete.

Let $|H_v| = 4$; it contains a Hamilton circuit since H_v is 2-connected. Then H_v must be a clique if it is a leaf-block. Hence we may suppose now that H_v is not a leaf-block. And every pair of cut vertices contained in H_v are not joined by a Hamilton path P_4 of H_v for otherwise, H has a P_6. Therefore, H_v has precisely two cut vertices u and v (see Fig. 4(a)) which are not next to each other along the 4-cycle of H_v. It is also clear that H_v is not a 4-cycle, for otherwise, $H + uv$ does not have a P_6 since H does not. So, $H_v = K_4 - e$ (see Fig. 4(a)). Now we may add a new edge e' joining $x \in N(v) - V(H_v)$ and u (see Fig. 4(a)) into H. But $H + e'$ does not have P_6 since H does not. Therefore H_v must be a clique.

Now we may assume that $|H_v| \geq 5$. Following the same arguments as were used to prove Lemma 4.1 (but not the corollary of it), we obtain that the structure of H_v is a book with common edge e. If $v \in e$, then all edges connecting neighbors of v not in H_v to u can be added to H. In the resulting graph, v is not a cut vertex any more. If $v \notin e$, then H has the above structure (see Fig. 4(b)) and contains P_6 as a subgraph, a contradiction. $\square$

According to the above lemmas, any connected P_6-saturated graph with order at least 6 is either a book or one of the following types, where B_n is the book with n triangles sharing one common edge, F_n is obtained from K_4 by gluing to the center of star S_n, T_n is the union of n triangles obtained by sharing one common vertex and T_n' is obtained from T_n by adding one more edge; $T_{i,j,k}$ is a rooted three-level tree.

Let G be a P_6-saturated graph (Fig. 5). Denote by $a, b, c, \alpha, \beta_1, \beta_2, \gamma, \delta$ the numbers of components isomorphic to the various structures $K_3, K_4, K_5, F_i, T_j, T_{j'}'$, trees and books in G for the remainder of this section. For any graph G, we denote by $r(G) = |E(G)| - |V(G)|$ the rank of G. Thus $r(K_4) = r(F_i) = 2$, $r(K_5) = 5$.

Lemma 4.3. *Let $m = 2n - 4$ and $n \geq 10$ be positive integers. There exists an (n, m) graph in $SAT(n, P_6)$ if and only if $n \equiv 1, 3 \pmod 5$.*

Proof. Let $n \equiv 1 \pmod 5$ and $m = 2n - 4$. We can construct an (n, m) graph $G = F_2 \cup \alpha K_5$, where $\alpha = (n - 6)/5$. Then $|V(G)| = 5\alpha + 6$ and $|E(G)| = 10\alpha + 8 = 2n - 4$. If $n \equiv 3 \pmod 5$, we build an (n, m) graph $G = 2K_4 \cup \alpha K_5$, where $\alpha = (n - 8)/5$. Then $|V(G)| = 5\alpha + 8$ and $|E(G)| = 10\alpha + 12 = 2n - 4$.

Next we shall prove by contradiction that there is no (n, m) graph in $SAT(n, P_6)$ with $m = 2n - 4$ if $n \not\equiv 1, 3 \pmod 5$. Suppose G is such a graph with components $H_1, H_2, \ldots, H_l$ and $h_1 \leq h_2 \leq \cdots \leq h_l$, where $h_i = |H_i|$. If $h_1 = 1$, then each component H_i would be K_5 for every $i \neq 1$. Therefore we have $n = 1 \pmod 5$, a contradiction. If $c_1 = 2$, the remaining components of G would be either K_4, K_5, F_i or a book B_j. Then we have

$$n = 2 + 4b + 5c + \sum_{i=1}^{\alpha} |V(F_i)| + \sum_{j=1}^{\delta} |V(B_j)|$$

and

$$m = 1 + 6b + 10c + \sum_{i=1}^{\alpha} (|V(F_i)| + 2) + \sum_{j=1}^{\delta} (2|V(B_j)| - 3),$$

where $|V(F_i)|, |V(B_j)| \geq 6$. Hence $4 = 2n - m = 3 + 2b + \sum_{i=1}^{\alpha}(|V(F_i)| - 2) + 3\delta$. Simplifying this we have

$$2b + \sum_{i=1}^{\alpha}(|V(F_i)| - 2) + 3\delta = 1.$$

It is easy to see that there is no integer solution to this equation.

So far we may suppose that $h_1 \geq 3$, and the remaining components of G are $K_3, K_4, K_5, F_i, T_j, T'_{j'}$, trees $T^{(k)}$ and books B_l. By counting the number of vertices and edges of each component, we obtain

$$n = 3a + 4b + 5c + \sum_{i=1}^{\alpha} |V(F_i)| + \sum_{j=1}^{\beta_1} |V(T_j)| + \sum_{j'=1}^{\beta_2} |V(T'_{j'})| + \sum_{k=1}^{\gamma} |V(T^{(k)})| + \sum_{l=1}^{\delta} |V(B_l)|$$

and

$$\begin{aligned} m = {} & 3a + 6b + 10c + \sum_{i=1}^{\alpha}(|V(F_i)| + 2) + \sum_{j=1}^{\beta_1} \frac{3(|V(T_j)| - 1)}{2} + \sum_{j'=1}^{\beta_2} \left(\frac{3|V(T'_{j'})|}{2} - 2 \right) \\ & + \sum_{k=1}^{\gamma}(|V(T^{(k)})| - 1) + \sum_{l=1}^{\delta}(2|V(B_l)| - 3), \end{aligned}$$

where $|V(F_i)|, |V(T_j)|, |V(T'_{j'})|, |V(B_l)| \geq 6$ and $|V(T^{(k)})| \geq 10$. Then we have

$$4 = 2n - m = 3a + 2b + \sum_{i=1}^{\alpha}(|V(F_i)| - 2) + \sum_{j=1}^{\beta_1} \frac{|V(T_j)| + 3}{2} + \sum_{j'=1}^{\beta_2} \left(\frac{|V(T'_{j'})|}{2} + 2 \right) + \sum_{k=1}^{\gamma}(|V(T^{(k)})| + 1) + 3\delta$$

which implies that $\beta_1 = \beta_2 = \gamma = 0$, and $\alpha \leq 1$. Plugging into the above equation we obtain $4 = 2b + 3(a + \delta) + \alpha(|V(F)| - 2)$. The integer solution to this equation is either $\alpha = 1$, $|V(F)| = 6$, $a = b = \delta = 0$ or $b = 2$, $a = \delta = \alpha = 0$. The first case implies that G is F_2 with six vertices and eight edges, a contradiction. For the latter case, G is the union of two K_4's and some K_5's, implying that $n = 8 + 5c$, contradicting the assumption that $n \not\equiv 1, 3 \pmod 5$. □

By a similar argument we have the following lemma.

Lemma 4.4. *If n is an integer and divisible by (5), then there is no (n, m) graph in $SAT(n, P_6)$ with $m = 2n - 2$ or $m = 2n - 1$.*

Proof. We prove this by contradiction. Suppose G is such an (n, m) graph in $SAT(n, P_6)$ with $2n - m \in \{1, 2\}$. Then $h_1 \geq 2$. If $h_1 = 2$, then $2n - m = 3 + 2b + \sum_{i=1}^{\alpha}(|V(F_i)| - 2) + 3\beta \geq 3$, a contradiction. Hence we may assume that $h_1 \geq 3$. Arguing as we did in the previous lemma we have

$$2n - m = 3a + 2b + \sum_{i=1}^{\alpha}(|V(F_i)| - 2) + \sum_{j=1}^{\beta_1} \frac{|V(T_j)| + 3}{2} + \sum_{j'=1}^{\beta_2} \left(\frac{|V(T'_{j'})|}{2} + 2 \right) + \sum_{k=1}^{\gamma}(|V(T^{(k)})| + 1) + 3\delta.$$

The constraint $2n - m \in \{1, 2\}$ implies that $b = 1$, $a = \alpha = \beta_1 = \beta_2 = \gamma = \delta = 0$ and hence G is the union of K_4 and some $K'_5 s$. Therefore we have $n = 4 + 5c$, contradicting the fact that $5|n$. □

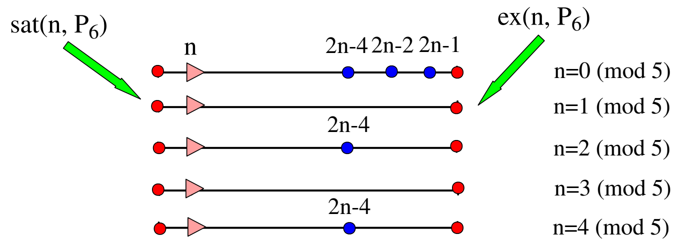
Lemma 4.5. *Let n be an integer that is at least (15); then there is a P_6-saturated (n, m) graph for all $\lceil \frac{9n}{10} \rceil \leq m \leq n + 5$.*

Proof. For those values of m in $[n, n+5]$, we can construct P_6-saturated graphs as follows: $T'_2 \cup T_{2,2,n-14}$, $K_2 \cup F_{n-6}$, F_{n-4}, $K_2 \cup K_4 \cup F_{n-10}$, $K_4 \cup F_{n-8}$, $B_4 \cup F_{n-10}$.

Let $n = 10k + i$, where $i = 0, 1, \ldots, 9$. We can construct an (n, m) graph in $SAT(n, P_6)$ as follows: $G_0 = (k-1)T_{2,2,2} \cup T_{2,2,2+i}$, and then $m = |E(G)| = 9(k-1) + 9 + i = 9k + i = \lceil \frac{9n}{10} \rceil$. Next we shall define an operation as follows: $T_{a,b,c} + T_{x,y,z} \Rightarrow T_{x,y,z+a+b+c+4}$. Under this operation we build a new P_6-saturated graph $T_{x,y,z+a+b+c+4}$ from a given P_6-saturated graph $T_{a,b,c} \cup T_{x,y,z}$ with one more edge. Continuing this process starting from G_0, we can end this process with $T_{2,2,n-8}$, which has $n - 1$ edges. Hence we find P_6-saturated graphs with m edges for $\lceil \frac{9n}{10} \rceil \leq m \leq n - 1$. □

Combining Lemma 4.1 through 4.5 we shall get the next main theorem concerning the edge spectrum of P_6. But this time we jump five steps at each induction process stage.

Theorem 4.1. *Let $n \geq 10$ be an integer. There is a P_6-saturated (n, m) graph with $(n, m) \neq (11, 14)$ if and only if m is in the following interval:*



where the triangle "△" means the existence of a P_6-saturated (n, n) graph only for $n \geq 15$, and a blue dot stands for a missing value.

Proof. The proof is by induction on n based on the following table, where $A = sat(n, P_6)$ and $B = ex(n, P_6)$. The initial results for $10 \leq n \leq 14$ are listed in the table. One exception in this table is $n = 11, m = 14$. There is no such kinds of P_6-saturated graphs, but for the next induction set we do have a $(16, 24)$-graph $4K_4$, which is a P_6-saturated graph. For the base case of the induction process, it is easy to check the graph listed in the table, while for those not listed in the table we can count all possible combinations of basic structures of P_6-saturated graphs for the corresponding (n, m)-values.

By the induction hypothesis, suppose we have the result for $n \leq 14$. By adding a complete graph K_5, we may get some partial results for $n + 5$. For the missing values on the interval $[A, B]$, we shall refer to Lemma 4.3 up to Lemma 4.5. So far we have covered the interval $[sat(n, P_6) + 10, B]$ since $ex(n, P_6) + 10 = B$. The remaining interval $[A, sat(n, P_6) + 9]$ will be covered by Lemma 4.6 since $sat(n, P_6) + 9 \leq (n + 5) + 5 = n + 10$ for any integer n.

n	$[AB]$	$SAT(n, P_6)$	n	$[A, B]$	$SAT(n, P_6)$
10	[9, 20]	9: $T_{2,2,2}$	11	[10, 20]	10: $T_{2,2,3}$
		10: $\emptyset$			11: $\emptyset$
		11: $K_2 \cup F_4$			12: $K_2 \cup F_5$
		12: F_6			13: F_7
		13: T_4'			14: $\emptyset$
		14: $K_2 \cup B_6$			15: $K_3 \cup 2K_4$
		15: $K_4 \cup B_4$			16: $2K_3 \cup K_5$
		16: $\emptyset$			17: $T_2' \cup K_5$
		17: B_8			18: $K_5 \cup F_2$
		18–19: $\emptyset$			19: B_9
		20: $2K_5$			20: $2K_5 \cup K_1$
12	[11, 21]	11: $T_{2,3,3}$	13	[12, 23]	12: $T_{3,3,3}$
		12: $4K_3$			13: $\emptyset$
		13: $F_6 \cup K_2$			14: $K_2 \cup F_7$
		14: F_8			15: F_9
		15: $F_2 \cup K_4 \cup K_2$			16: $F_3 \cup K_4 \cup K_2$
		16: $K_4 \cup F_4$			17: $K_4 \cup F_5$
		17: $F_2 \cup B_4$			18: T_6
		18: $K_2 \cup B_8$			19: $F_2 \cup B_5$
		19: $K_3 \cup K_4 \cup K_5$			20: $K_2 \cup B_9$
		20: $\emptyset$			21: $K_4 \cup B_7$
		21: $2K_5 \cup K_2$			22: $2K_4 \cup K_5$
					23: $K_3 \cup 2K_5$
14	[13, 19]	13: $T_{3,3,4}$	14	[20, 26]	20: $F_3 \cup B_5$
		14: $\emptyset$			21: $F_5 \cup K_5$
		15: $K_2 \cup F_8$			22: $K_2 \cup B_{10}$
		16: F_{10}			23: $K_4 \cup B_8$
		17: $T_2' \cup F_4$			24: $\emptyset$
		18: $K_3 \cup T_5$			25: B_{12}
		19: T_6'			26: $2K_5 \cup K_4$

□

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