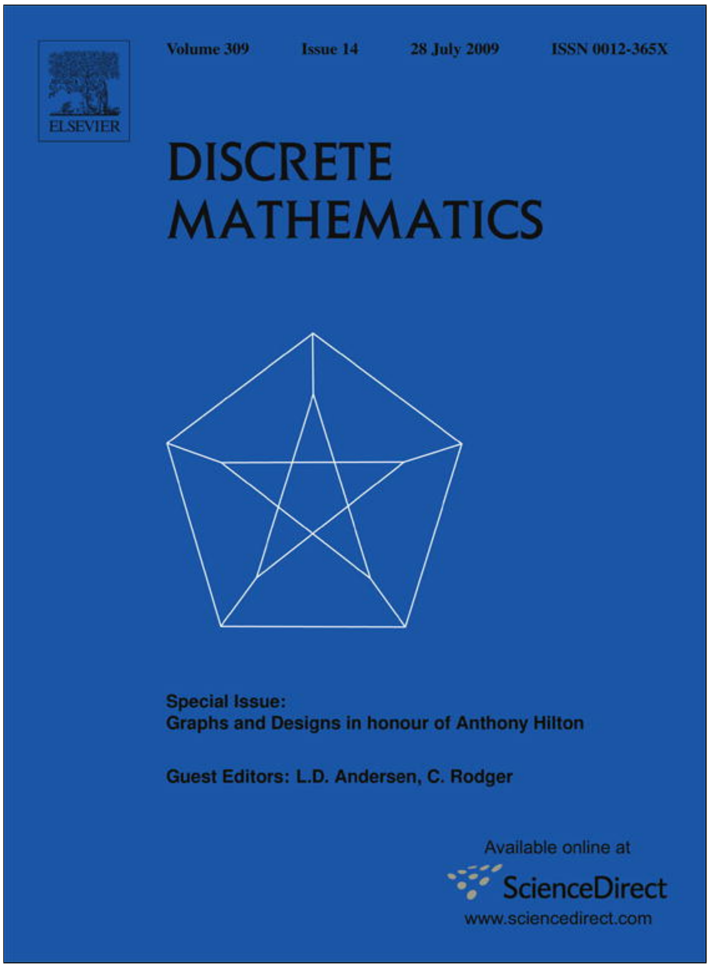

Volume 309
Issue 14
28 July 2009
ISSN 0012-365X
ELSEVIER
DISCRETE MATHEMATICS
Special Issue:
Graphs and Designs in honour of Anthony Hilton
Guest Editors: L.D. Andersen, C. Rodger
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 309 (2009) 4682–4689

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Flows, flow-pair covers and cycle double covers

Dezheng Xie [a], Cun-Quan Zhang [b,*]

[a] *Department of Mathematics, Chongqing University, Chongqing, 400041, China*

[b] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA*

ARTICLE INFO

Article history:
Received 13 December 2006
Accepted 28 May 2008
Available online 15 July 2008

Keywords:
Integer flow
Flow cover
Cycle double cover

ABSTRACT

In this paper, some earlier results by Fleischner [H. Fleischner, Bipartizing matchings and Sabidussi's compatibility conjecture, Discrete Math. 244 (2002) 77–82] about edge-disjoint bipartizing matchings of a cubic graph with a dominating circuit are generalized for graphs without the assumption of the existence of a dominating circuit and 3-regularity. A pair of integer flows (D, f_1) and (D, f_2) is an (h, k)-flow parity-pair-cover of G if the union of their supports covers the entire graph; f_1 is an h-flow and f_2 is a k-flow, and $E_{f_1=\text{odd}} = E_{f_2=\text{odd}}$. Then G admits a nowhere-zero 6-flow if and only if G admits a $(4, 3)$-flow parity-pair-cover; and G admits a nowhere-zero 5-flow if G admits a $(3, 3)$-flow parity-pair-cover. A pair of integer flows (D, f_1) and (D, f_2) is an (h, k)-flow even-disjoint-pair-cover of G if the union of their supports covers the entire graph, f_1 is an h-flow and f_2 is a k-flow, and $E_{f_i=\text{even}, f_i\neq 0} \subseteq E_{f_j=0}$ for each $\{i, j\} = \{1, 2\}$. Then G has a 5-cycle double cover if G admits a $(4, 4)$-flow even-disjoint-pair-cover; and G admits a $(3, 3)$-flow parity-pair-cover if G has an orientable 5-cycle double cover.

1. Introduction

Assume that C is a dominating circuit of a cubic graph G. A matching M of G is bipartizing with respect to C if $M \cap E(C) = \emptyset$, $V(G) - V(C) \subseteq V(M)$ and either the cubic graph homeomorphic to $G - E(M)$ is bipartite or $G - E(M)$ is 2-regular. The concept of bipartizing matching was introduced in [3,5]. *If G contains a pair of edge-disjoint bipartizing matchings with respect to a dominating circuit C, then it was proved in* [3] *that G has a 5-cycle double cover and was proved in* [5] *that G admits a nowhere-zero 5-flow.* Applying an observation by Tutte [15] (Lemma 2.1) that *a cubic graph admits a nowhere-zero 3-flow if and only if the graph is bipartite*, one can see that the statement that a cubic graph contains a pair of edge-disjoint bipartizing matchings with respect to a dominating circuit implies the existence of a pair of 3-flows with a special arrangement of the flow values in the graph. This relation inspires a further investigation of flow coverings of graphs in this paper.

The concept of flow-pair covering, introduced in many publications, has been used as one of the major techniques in the studies of integer flows and cycle covers. A pair of integer flows (D, f_1) and (D, f_2) is a cover of a graph G if $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$. The following lemma addresses the relationship between flow covers and nowhere-zero flows.

Lemma 1.1. *A graph G admits a nowhere-zero hk-flow if and only if G admits an h-flow (D, f_1) and a k-flow (D, f_2) such that* $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.

The "*if*" part of Lemma 1.1 was applied in the proofs of many celebrated results in the integer flow theory [7,11]. If a graph is covered by two flows, then it admits a nowhere-zero flow as their "product" (by Lemma 1.1). If a graph G is covered by two flows in a certain special arrangement, then it is possible that G may admit a nowhere-zero flow stronger/better

* Corresponding author.
E-mail addresses: xdzwp568@ctbu.edu.cn (D. Xie), cqzhang@math.wvu.edu (C.-Q. Zhang).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.05.056

than their "product". This was initially discussed by Fleischner in [4,5]. He proved that if a cubic graph G with a dominating circuit C has a pair of 3-flows $\{(D, f_1), (D, f_2)\}$ with $E_{f_1=\text{odd}} = E_{f_2=\text{odd}} = E(C)$ covering G, then G admits a nowhere-zero 5-flow and has a circuit double cover. These pioneer results by Fleischner [4,5] are to be generalized in this paper for graphs without requiring the existence of a dominating circuit and 3-regularity.

Some famous theorems in integer flow theory were proved by applying Lemma 1.1: the 6-flow theorem [11] is proved via the product of a pair of a 3-flow and a 2-flow, the 4-flow theorem for 4-edge-connected graphs [7] is proved via the product of a pair of 2-flows, and the 8-flow theorem [7] is proved via the product of three 2-flows.

From all those approaches, one might propose that the famous 5-flow conjecture by Tutte [16] could be approached by covering a bridgeless graph with a pair of a 2-flow and a $\frac{5}{2}$-flow ($\frac{5}{2}$-flow is defined as a circular flow, see [6,20] for definition). However, this approach does not work for cubic graphs since the support of a $\frac{5}{2}$-flow must be odd-5-edge-connected. A result in this paper raises a new hope that the flow-pair covering approach introduced by Jaeger and Seymour [7,11] may still work for the 5-flow conjecture although 5 is a prime number.

2. Notations and useful lemmas

For notations not defined here see [1].

A circuit is a connected 2-regular graph, while a cycle is a graph with even degree for every vertex. An edge e is a bridge of a graph G if the removal of e increases the number of components. A graph G is *odd-$(2k+1)$-edge-connected* if the size of every odd edge cut is at least $2k + 1$.

Let G be a graph such that the degree of every vertex is either 2 or 3. The cubic graph homeomorphic to G is denoted by $\overline{G}$ and is called the *underlying graph* of G.

Let Z be the set of all integers. Let $G = (V, E)$ be a graph and let k be a positive integer. An ordered pair (D, ϕ) is called an *integer k-flow* of G if $D = (V, A)$ is an orientation of G and $\phi : A \mapsto Z$ is an assignment of flow values such that for every vertex v,

$$\sum_{e\in E^+(v)} \phi(e) = \sum_{e\in E^-(v)} \phi(e)$$

and

$$|\phi(e)| < k \quad \text{for all } e \in E(G)$$

where $E^+(v)$ (resp. $E^-(v)$) is the set of all edges of G under the orientation D with tail v (resp. head v). We say that (D, ϕ) is a *nowhere-zero flow* if $\phi(e) \neq 0$ for all $e \in E(G)$. This concept was introduced by Tutte [16], and the theory of nowhere-zero flows generalizes map-coloring theorems and conjectures for planar graphs to general graphs. Major open problems in this area are Tutte's celebrated 3-, 4-, and 5-flow conjectures. Interested readers are referred to [8,13] for the main ideas of this subject and to [18,19] for in-depth accounts.

Let $G = (V, E)$ be a graph and let k be a positive integer. An ordered pair (D, ϕ) is called a *modular k-flow* of G if $D = (V, A)$ is an orientation of G and $\phi : A \mapsto Z$ is an assignment of flow values such that for every vertex v,

$$\sum_{e\in E^+(v)} \phi(e) \equiv \sum_{e\in E^-(v)} \phi(e) \pmod k.$$

We say that (D, ϕ) is *nowhere-zero* if $\phi(e) \not\equiv 0 \pmod k$ for all $e \in E(G)$.

For a flow (D, f) of G, the *support* of f is the set $\text{supp}(f) = \{e \in E(G) : f(e) \neq 0\}$. Denote $E_{f=t} = f^{-1}(t)$, $E_{f=\pm t} = f^{-1}(t) \cup f^{-1}(-t)$, $E_{f=\text{odd}} = \{e \in E(G) : f(e) \equiv 1 \pmod 2\}$, $E_{f=\text{even}} = \{e \in E(G) : f(e) \equiv 0 \pmod 2\}$, and $E_{f=\text{even}, f\neq 0} = E_{f=\text{even}} \cap \text{supp}(f)$.

A family $\mathcal{C}$ of cycles of a graph G is a *cycle cover* of G if $E(G) = \bigcup_{C\in\mathcal{C}} E(C)$. A cycle cover $\mathcal{C}$ is a *k-cycle cover* if $|\mathcal{C}| = k$. A cycle cover $\mathcal{C}$ is a *double cover* of G if every edge of G is contained in precisely two members of $\mathcal{C}$ [10,14]. A cycle double cover $\mathcal{C}$ is a *k-cycle double cover* if $|\mathcal{C}| \leq k$.

A k-cycle double cover $\{C_1, \ldots, C_k\}$ of G is *orientable* if each C_i has an Eulerian orientation such that each edge e of G is covered by two cycles C_i and C_j with opposite directions.

The following are some fundamental lemmas in the theory of integer flow, which are to be used frequently in the remaining part of the paper.

Lemma 2.1 (*Tutte [15]*). *A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.*

Lemma 2.2 (*Tutte [17]*). *A graph admits a nowhere-zero 2-flow if and only if it is a cycle.*

Lemma 2.3 (*Tutte [17]*). *Let (D, f) be an integer flow of G. Then $E_{f=\text{odd}}$ induces a cycle of G.*

Lemma 2.4 (*Tutte [15]*). *If a graph G admits a modular k-flow (D, f), then G admits a integer k-flow (D, f') such that*

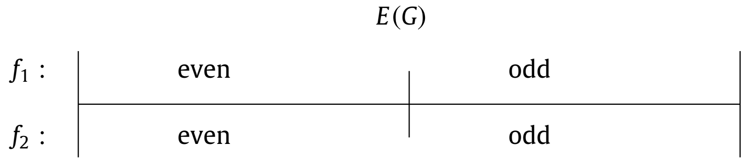
$$f(e) \equiv f'(e) \pmod k \quad \text{for all } e \in E(G).$$

Lemma 2.5 (*[15,12]*). *Let H be a subgraph of G. Then G admits a nowhere-zero 4-flow (D, f) with $E_{f=\text{even}} = E(H)$ if and only if G has a 2-cycle cover with the edges of H covered twice and all other edges covered once.*

3. Flow-pair coverings

Definition 3.1. Let G be a graph and h, k be two integers. Assume that (D, f_1) is an integer h-flow of G and (D, f_2) is an integer k-flow of G. If $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$ and $E_{f_1=\text{odd}} = E_{f_2=\text{odd}}$, then $\{(D, f_1), (D, f_2)\}$ is called an (h, k)-flow *parity-pair-cover* of G.

The distribution of edges with weights of different parity of an (h, k)-flow parity-pair-cover is illustrated in the following chart-like figure.



Definition 3.2. Let G be a graph and h, k be two integers. Assume that (D, f_1) is an integer h-flow of G and (D, f_2) is an integer k-flow of G. If

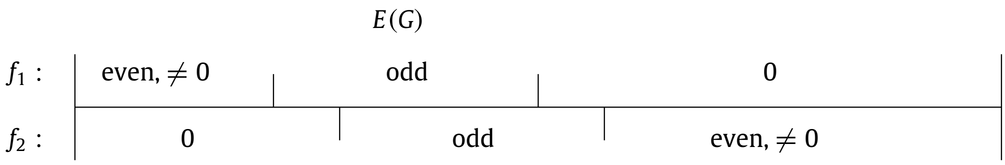
$$\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$$

and

$$E_{f_1=\text{even}, f_1 \neq 0} \subseteq E_{f_2=0}, \; E_{f_2=\text{even}, f_2 \neq 0} \subseteq E_{f_1=0},$$

then $\{(D, f_1), (D, f_2)\}$ is called *an* (h, k)*-flow even-disjoint-pair-cover of* G. (That is, $E(G)$ has a decomposition $\{A, B, C\}$ where $A = E_{f_1=\text{even}, f_1 \neq 0}$, $B = E_{f_2=\text{even}, f_2 \neq 0}$, $C = E_{f_1=\text{odd}} \cup E_{f_2=\text{odd}}$.)

The distribution of edges with weights of different parity of an (h, k)-flow even-disjoint-pair-cover is illustrated in the following chart-like figure.



Definition 3.3. Let G be a graph and h, k be two integers. Assume that (D, f_1) is an integer h-flow of G and (D, f_2) is an integer k-flow of G. If $\{(D, f_1), (D, f_2)\}$ is both an (h, k)-flow parity-pair-cover and an (h, k)-flow even-disjoint-pair-cover, then $\{(D, f_1), (D, f_2)\}$ is called *an* (h, k)*-flow strong parity-pair-cover of* G. (That is, the pair of integer flows $\{(D, f_1), (D, f_2)\}$ satisfies the descriptions of both Definitions 3.1 and 3.2.)

Most theorems in next sections are generalizations of some earlier results of Fleischner [4,5] for cubic graphs with respect to dominating circuits.

4. Nowhere-zero flows and flow covers

Theorem 4.1. *Let G be a graph. Then G admits a nowhere-zero* 6*-flow if and only if G admits a* $(4, 3)$*-flow parity-pair-cover.*

Proof. $\Leftarrow$: Assume that $\{(D, f_1), (D, f_2)\}$ is a $(4, 3)$-flow parity-pair-cover of G. Table 1 verifies that $(D, \frac{3f_1}{2} + \frac{f_2}{2})$ is a nowhere-zero 6-flow of G.

$\Rightarrow$: Assume that (D, f_3) is a nowhere-zero 6-flow of G. Let (D, f_4) be an integer 2-flow of G with

$$\text{supp}(f_4) = E_{f_3=\text{odd}}.$$

Table 1

f_1	f_2	$\frac{3f_1}{2} + \frac{f_2}{2}$
0	± 2	± 1
± 2	0	± 3
± 2	± 2	± 2 or ± 4
± 1	± 1	± 2 or ± 1
± 3	± 1	± 5 or ± 4

By definition, $(D, \frac{f_3+f_4}{2})$ is a modular 3-flow of G. Applying Lemma 2.4, let (D, f_5) be an integer 3-flow of G with

$$f_5 \equiv \frac{f_3+f_4}{2} \pmod 3.$$

Note that

$$\text{supp}(f_4) \cup \text{supp}(f_5) = E(G)$$

since the transition from a modular k-flow to a k-flow yields the same support.

Let (D, f_6) be an integer 2-flow of G with $\text{supp}(f_6) = E_{f_5=odd}$. Then $(D, f_7 = f_6 + 2f_4)$ is an integer 4-flow of G with

$$\text{supp}(f_7) \cup \text{supp}(f_5) = E(G)$$

and

$$E_{f_7=\text{odd}} = \text{supp}(f_6) = E_{f_5=\text{odd}}. \quad \blacksquare$$

By a similar method, we can also prove that a graph admits a nowhere-zero 8-flow if and only if the graph admits a (4, 4)-flow parity-pair-cover. Since it is known that every bridgeless graph admits a nowhere-zero 6-flow [11], we omit the corresponding proof for 8-flow.

Theorem 4.2. *Let G be a graph. Then G admits a nowhere-zero 4-flow if and only if G admits a (3, 3)-flow strong parity-pair-cover.*

Proof. $\Rightarrow$: Let (D, f) be a nowhere-zero 4-flow. Let (D, f_1) be an integer 2-flow with

$$\text{supp}(f_1) = E_{f=\text{odd}}.$$

Then

$$\left(D, f_2 = \frac{f+f_1}{2}\right)$$

is a 3-flow. Let (D, f_3) be an integer 2-flow of G with

$$\text{supp}(f_3) = E_{f_2=\text{odd}}.$$

Then

$$\left\{\left(D, \frac{f_1+f_3}{2}\right), \left(D, \frac{f_1-f_3}{2}\right)\right\}$$

is a (3, 3)-flow strong parity-pair-cover of G.

$\Leftarrow$: Assume that $\{(D, f'), (D, f'')\}$ is a (3, 3)-flow strong parity-pair-cover of the graph G. Then $(D, \frac{3}{2}f' + \frac{1}{2}f'')$ is a nowhere-zero 4-flow of G. See Table 2. $\blacksquare$

Table 2

f'	f''	$\frac{3f'}{2} + \frac{f''}{2}$
0	± 2	± 1
± 2	0	± 3
± 1	± 1	± 2 or ± 1

Theorem 4.3. *Let G be a graph. If G admits a (3, 3)-flow parity-pair-cover then G admits a nowhere-zero 5-flow.*

Proof. Assume that $\{(D, f_1), (D, f_2)\}$ is a (3, 3)-flow parity-pair-cover of G. Then $(D, \frac{3f_1}{2} + \frac{f_2}{2})$ is a nowhere-zero 5-flow of G. See Table 3. $\blacksquare$

Table 3

f_1	f_2	$\frac{3f_1}{2} + \frac{f_2}{2}$
0	± 2	± 1
± 2	0	± 3
± 2	± 2	± 2 or ± 4
± 1	± 1	± 2 or ± 1

In contrast to Theorems 4.1 and 4.3 is not an "*if and only if* " statement: the other direction is proposed as a conjecture (Conjecture 8.1).

Furthermore, one part of both Theorems 4.1 and 4.3 can be further generalized as follows:

In general, if G admits an $(h, 3)$-flow parity-pair-cover, then G admits a nowhere-zero j-flow where

$$j = \frac{3h+1}{2} \quad \text{if } h \text{ is odd,}$$

$$\text{and} \quad j = \frac{3h}{2} \quad \text{if } h \text{ is even.}$$

This general result is not posed as a theorem because we already know the existence of a nowhere-zero 6-flow for any bridgeless graph [11].

5. Flow parity-pair-covers and even-disjoint-pair-covers

Theorem 5.1. *Let G be a graph. Then G admits a $(3, 3)$-flow parity-pair-cover $\{(D, f_1), (D, f_2)\}$ if and only if G admits a $(3, 3)$-flow even-disjoint-pair-cover $\{(D, f'), (D, f'')\}$. Furthermore,*

$$E_{f_1=\pm 1} = E_{f_2=\pm 1} = E_{f'=1} \Delta E_{f''=1}.$$

Proof. $\Rightarrow$: Assume that G admits a $(3, 3)$-flow parity-pair-cover $\{(D, f_1), (D, f_2)\}$.

Then

$$\left\{(D, f') = \left(D, \frac{f_1+f_2}{2}\right), (D, f'') = \left(D, \frac{f_1-f_2}{2}\right)\right\}$$

is a $(3, 3)$-flow even-disjoint-pair-cover of G: for each $e \in E_{f_1=\pm 1} = E_{f_2=\pm 1}$, either $f'(e) = 1$ and $f''(e) = 0$, or $f'(e) = 0$ and $f''(e) = 1$; for each $e \notin E_{f_1=\pm 1} = E_{f_2=\pm 1}$, $f'(e) + f''(e) = \pm 2$.

$\Leftarrow$: Assume that the graph G admits a $(3, 3)$-flow even-disjoint-pair-cover $\{(D, f^\star), (D, f^{\star\star})\}$. Then

$$\left\{(D, f_3) = \left(D, f^\star + f^{\star\star}\right), (D, f_4) = \left(D, f^\star - f^{\star\star}\right)\right\}$$

is a $(3, 3)$-flow parity-pair-cover of G. ■

6. Cycle double covers

Theorem 6.1. *Let G be a graph. Then G admits a $(4, 4)$-flow even-disjoint-pair-cover $\{(D, f_1), (D, f_2)\}$ if and only if G has a 5-cycle double cover (which contains $E_{f_1=\text{odd}} \Delta E_{f_2=\text{odd}}$ as a member).*

Proof. $\Rightarrow$: Assume that $\{(D, f'), (D, f'')\}$ is a $(4, 4)$-flow even-disjoint-pair-cover of G. Consider $A = E_{f'=\text{even}, f'\neq 0}$, $B = E_{f''=\text{even}, f''\neq 0}$, and $C = E_{f'=\text{odd}} \cup E_{f''=\text{odd}}$, as a partition of $E(G)$. By Lemma 2.5, supp(f') has a 2-cycle cover, and so does supp(f''). The set of these four cycles covers each edge e twice if $e \in A \cup B = E(G) \setminus C$ or $e \in E_{f'=\text{odd}} \cap E_{f''=\text{odd}}$, and once if $e \in E_{f'=\text{odd}} \Delta E_{f''=\text{odd}}$.

$\Leftarrow$: Assume that $\{C_1, \ldots, C_5\}$ is a 5-cycle double cover of G. Let (D, f_i) be an integer 2-flow of G with supp$(f_i) = E(C_i)$ for each $i \in \{1, 2, 3, 4\}$. Let $C_{ij} = C_i \Delta C_j$ and (D, f_{ij}) be an integer 2-flow of G with supp$(f_{ij}) = E(C_{ij})$ for each $\{i, j\} \subset \{1, 2, 3, 4\}$.

Then $\{(D, f_{12} + 2f_2), (D, f_{34} + 2f_4)\}$ is a $(4, 4)$-flow even-disjoint-pair-cover with

$$E_{f_{12}+2f_2=\text{odd}} \Delta E_{f_{34}+2f_4=\text{odd}} = E(C_5). \quad ■$$

Theorem 6.2. *If G has an orientable 5-cycle double cover, then G has a $(3, 3)$-flow even-disjoint-pair-cover.*

Proof. Assume that $\{C_1, \ldots, C_5\}$ is an orientable 5-cycle double cover of G with each C_μ associated with an Eulerian orientation D_μ, $\mu = 1, \ldots, 5$, such that for each edge e, both orientations are used in the corresponding C_μ's. Let (D_μ, f_μ) be a non-negative integer 2-flow of G with support C_μ. Let D be an arbitrary orientation of G, and let (D, f'_μ) be an integer 2-flow of G with $f'_\mu(e) = f_\mu(e)$ if the edge e has the same orientation of D and D_μ, and $f'_\mu(e) = -f_\mu(e)$ if the edge e has the opposite orientations of D and D_μ. Thus, $\{(D, f'_1 - f'_2), (D, f'_3 - f'_4)\}$ is a $(3, 3)$-flow even-disjoint-pair-cover of G. ■

Note that the existence of 5-cycle double covers and orientable 5-cycle double covers are conjectured for all bridgeless graphs.

Conjecture 6.3 (*Preissmann [9] and Celmins [2]*). *Every bridgeless graph has a 5-cycle double cover.*

Conjecture 6.4 (*Jaeger [8]*). *Every bridgeless graph has an orientable 5-cycle double cover.*

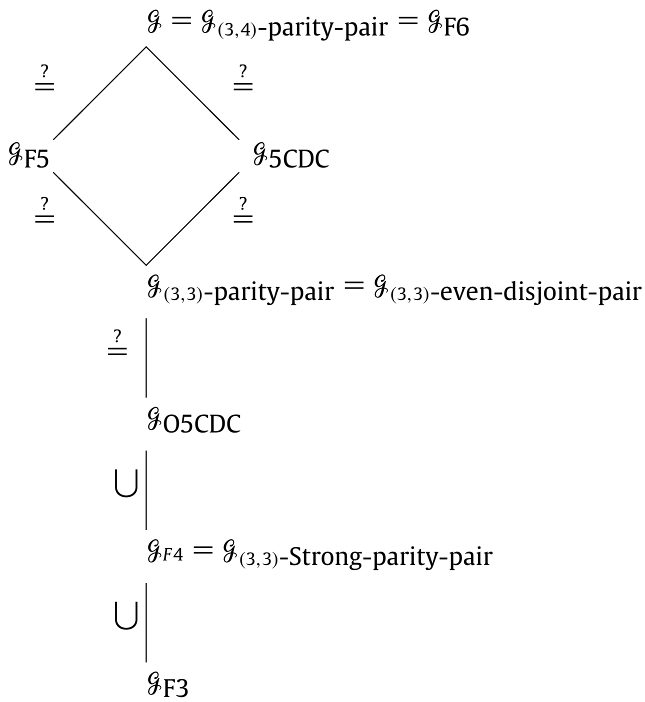
The following proposition was observed by one of the referees of this paper without applying the 4-color theorem.

Proposition 6.5. *Every planar graph has a $(3, 3)$-flow parity-pair-cover.*

Proof. Since every planar bridgeless graph has a proper 5-face-coloring, and thus has an orientable 5-cycle double cover (each cycle is defined by the face boundaries of a color class). Now apply Theorem 6.2 and subsequently Theorem 5.1. ■

7. Inclusion relations

By the theorems above, we have the following relations (as a poset):

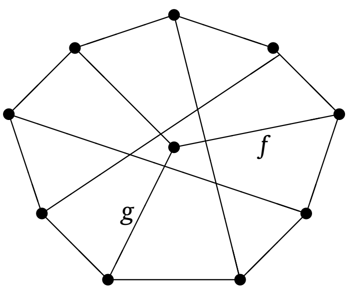


Some explanations about the poset:

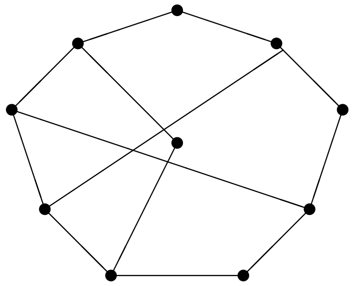
- $\mathcal{G}$ is the family of all bridgeless graphs;
- $\mathcal{G}_{Fk}$ is the family of graphs admitting nowhere-zero k-flows;
- $\mathcal{G}_{(h,k)\text{-parity-pair}}$ is the family of graphs admitting (h, k)-flow parity-pair-covers;
- $\mathcal{G}_{(h,k)\text{-even-disjoint-pair}}$ is the family of graphs admitting (h, k)-flow even-disjoint-pair-covers;
- $\mathcal{G}_{(h,k)\text{-strong-parity-pair}}$ is the family of graphs admitting (h, k)-flow strong parity-pair-covers;
- $\mathcal{G}_{5CDC}$ is the family of graphs admitting 5-cycle double covers;
- $\mathcal{G}_{O5CDC}$ is the family of graphs admitting orientable 5-cycle double covers;
- The symbol "$\bigcup$" indicates that one family of graphs is a proper subset of another family;
- The symbol "$\stackrel{?}{=}$" indicates that one family of graphs is a subset of another family, and that is further conjectured that they are the same.

For example, by Theorems 5.1 and 6.1, $\mathcal{G}_{O5CDC} \subseteq \mathcal{G}_{(3,3)\text{-parity-pair}}$ and it is conjectured that $\mathcal{G}_{O5CDC} = \mathcal{G}_{(3,3)\text{-parity-pair}}$ (Conjecture 6.3). Note that $\mathcal{G}_{F4} \subset \mathcal{G}_{O5CDC}$ since the Petersen graph is in $\mathcal{G}_{O5CDC} - \mathcal{G}_{F4}$.

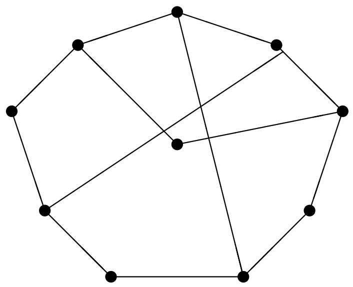
The following figures describe a pair of 3-flows in P_{10} which form a $(3, 3)$-flow parity-pair-cover.



The Petersen graph P_{10}



A 3*-flow* (D, f_1) *of* $P_{10} - f$*:*
$f_1(e) = \pm 1$ *for each edge e belonging to the dominating circuit,*
$f_1(e) = \pm 2$ *for all other edges.*



A 3*-flow* (D, f_2) *of* $P_{10} - g$*:*
$f_2(e) = \pm 1$ *for each edge e belonging to the dominating circuit,*
$f_2(e) = \pm 2$ *for all other edges.*

8. Open problems

Conjecture 8.1. *Every bridgeless cubic graph G admits a* $(3, 3)$*-flow parity-pair-cover.*

With Tutte's 5-flow Conjecture, the following is a weak version of Conjecture 8.1.

Conjecture 8.2. *If G admits a nowhere-zero* 5*-flow, then G admits a* $(3, 3)$*-flow parity-pair-cover.*

By Theorems 4.2 and 4.3, the $(3, 3)$-flow parity-pair-cover property lies between the properties of 4-flow and 5-flow.

Problem 8.3. Can we find a rational number $r : 4 \leq r \leq 5$ such that the $(3, 3)$-flow parity-pair-cover property is equivalent to the circular r-flow property?

The following conjecture is equivalent to the 5-cycle double cover conjecture (because of Theorem 6.1).

Conjecture 8.4. *Every graph G admits an* $(4, 4)$*-flow even-disjoint-pair-cover* $\{(D, f_1), (D, f_2)\}$.

Acknowledgments

Second author was supported in part by the National Security Agency under Grant H98230-05-1-0080 and by a WV RCG grant.

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