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A note on Berge–Fulkerson coloring

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ABSTRACT

The Berge–Fulkerson Conjecture states that *every cubic bridgeless graph has six perfect matchings such that every edge of the graph is contained in exactly two of these perfect matchings*. In this paper, a useful technical lemma is proved that *a cubic graph G admits a Berge–Fulkerson coloring* if and only if *the graph G contains a pair of edge-disjoint matchings M_1 and M_2* such that (i) *$M_1 \cup M_2$ induces a 2-regular subgraph of G* and (ii) *the suppressed graph $\overline{G \setminus M_i}$, the graph obtained from $G \setminus M_i$ by suppressing all degree-2-vertices, is 3-edge-colorable for each $i = 1, 2$*. This lemma is further applied in the verification of Berge–Fulkerson Conjecture for some families of non-3-edge-colorable cubic graphs (such as, *Goldberg snarks, flower snarks*).

1. Introduction

Proper 3-edge-coloring of cubic graphs has been extensively studied due to its equivalency to the 4-color problem of planar graphs. However, not every cubic graph is 3-edge-colorable. The following is one of the most famous conjectures in graph theory.

Conjecture 1.1 (*Berge, Fulkerson, [9], or see [13,1,5]*). *Every 2-connected cubic graph has a collection of six perfect matchings that together cover every edge exactly twice.*

Although the statement of the conjecture is very simple, the solution has eluded many mathematicians over four decades and remains beyond the horizon. Due to the lack of appropriate techniques, not many partial results have been proved. In this paper, we would like to introduce some techniques for this problem and verify the conjecture for some families of cubic graphs.

The problem of matching covering is one of the major subjects in graph theory because of its close relation with problems of cycle cover, integer flow and other problems. Many generalizations and variations of the Berge–Fulkerson Conjecture have already received extensive attention, and some partial results have been achieved.

An *r-graph G* is an r-regular graph such that $|(X, V(G) \setminus X)_G| \geq r$, for every non-empty vertex subset $X \subseteq V(G)$ of odd order where $(X, V(G) \setminus X)_G = \{e = uv \in E(G) : u \in X, v \notin X\}$. It was proved by Edmonds [6] (also see [13]) that, *for a given r-graph G, there is an integer k (a function of G) such that G has a family of perfect matchings which covers each edge precisely k times.* Motivated by this result, Seymour [13] further conjectured that *every r-graph has a Berge–Fulkerson coloring.*

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Note that the complement of a perfect matching in a cubic graph is a 2-factor. The Berge–Fulkerson Conjecture is equivalent to the conjecture that *every bridgeless cubic graph has a family of six cycles such that every edge is covered precisely four times.* It was proved by Bermond, Jackson and Jaeger [2] that *every bridgeless graph has a family of seven cycles such that every edge is covered precisely four times;* and Fan [7] proved that *every bridgeless graph has a family of ten cycles such that every edge is covered precisely six times.*

The relation between Berge–Fulkerson coloring and shortest cycle cover problems has been investigated by Fan and Raspaud [8]. They proved that *if the Berge–Fulkerson Conjecture is true, then every bridgeless graph has a family of cycles that covers all edges and has the total length at most* $\frac{22}{15}|E(G)|$. The famous cycle double cover conjecture (Szekeres [15], Seymour [14]) would be verified if one could find a cycle cover of every cubic graph with total length at most $\frac{21}{15}|E(G)|$ (Jamshy and Tarsi [12]).

A non-3-edge-colorable, bridgeless, cyclically 4-edge-connected, cubic graph is called a *snark*. In this paper, we verify the conjecture for the families of Goldberg snarks and flower snarks.

2. Notation

Most standard terminology and notation can be found in [3] or [16].

Let G be a cubic graph. The graph $2G$ is obtained from G by duplicating every edge to become a pair of parallel edges.

For a cubic graph G, a *Berge–Fulkerson coloring* is a mapping $c : E(2G) \mapsto \{1, 2, \dots, 6\}$ such that every vertex of $2G$ is incident with edges colored with all six colors.

A *circuit* is a connected 2-regular subgraph. A *cycle* is the union of a set of edge-disjoint circuits. An edge is called a *bridge* if it is not contained in any circuit of the graph.

Let $G = (V, E)$ be a graph. The *suppressed graph*, denote by $\overline{G}$, is the graph obtained from G by suppressing all degree-2-vertices. In this paper, it is possible that some graph G may contain a 2-regular component C, and therefore, the 2-regular subgraph C corresponds to a *vertexless loop* in the suppressed graph $\overline{G}$.

A *vertexless loop* is a special case in this paper that is not usually seen in most literature. For graphs with vertexless loops, we may further extend some popularly used terminology. For example, the degree of a vertex is defined the same as usual. Therefore, a graph is cubic if the degree of every vertex is 3 while vertexless loops are allowed. A 3-edge-coloring of a cubic graph is a mapping $c : E(G) \mapsto \{1, 2, 3\}$ such that every vertex is incident with edges colored with all three colors. Hence, those vertexless loops may be colored with any color.

3. A technical lemma

In this section, we provide a useful technical lemma. This lemma is further applied in the proofs of other results of the paper.

Lemma 3.1 ([17]). *Given a cubic graph G admits a Berge–Fulkerson coloring if and only if there are two edge-disjoint matchings M_1 and M_2 such that each $\overline{G \setminus M_i}$ is 3-edge-colorable for $i = 1, 2$ and $M_1 \cup M_2$ forms a cycle in G.*

Remark 1. For each $\{i, j\} = \{1, 2\}$, the suppressed graph $\overline{G \setminus M_i}$ may contain some trivial components (vertexless loops). As we discussed in the previous section, every vertexless loop of $\overline{G \setminus M_i}$ corresponds to a 2-regular component C of $G \setminus M_i$. Since $M_i \bigcup M_j$ is a cycle, every vertex of C must be incident with edges of both M_i and M_j. Hence, the edges of the circuit C must be alternately in M_j and $E(G) \setminus \{M_i \bigcup M_j\}$. It is easy to see that every 2-regular component of $G \setminus M_i$ must be an even length circuit.

Proof. "$\Rightarrow$": We only pay attention to non-3-edge-colorable graphs, since every 3-edge-colorable cubic graph trivially satisfies the conjecture with $M_1 \bigcup M_2 = \emptyset$. Suppose G admits a Berge–Fulkerson coloring $c : E(2G) \mapsto \{a_1, b_1, c_1, a_2, b_2, c_2\}$. Let G_1 be the cubic subgraph of $2G$ induced by edges colored with $\{a_1, b_1, c_1\}$ and let G_2 be the subgraph of $2G$ induced by edges colored with $\{a_2, b_2, c_2\}$. If there is no parallel edge in both G_1 and G_2, then each of G_1 and G_2 is isomorphic to G. Hence G is 3-edge-colorable, so we may assume that there are parallel edges in G_1 and G_2 and let E_p be the set of edges of G corresponding to parallel edges in either G_1 or G_2.

Let M_j be the set of edges e of G such that e corresponds to a parallel edge of G_j for each $j = 1, 2$ and $E_p = M_1 \cup M_2$. Since G_j is cubic, M_j is a matching and $M_1 \cap M_2 = \emptyset$. Furthermore, each vertex incident with an edge of M_i must be incident with an edge of M_j. Hence, E_p is a set of edge-disjoint even circuits.

Now, for each $\{i, j\} = \{1, 2\}$, we may color each edge $e \in M_j$ with the color $\{a_j, b_j, c_j\} - \{x_j, y_j\}$ where x_j, y_j are colors used for the parallel edges corresponding to e in the cubic graph G_j. So, the resulting coloring is a proper 3-edge-coloring of the suppressed cubic graph $\overline{G \setminus M_i}$ for each $\{i, j\} = \{1, 2\}$. (Note that the cubic graph $\overline{G \setminus M_i}$ is also obtained from G_j by replacing every pair of parallel edges with a single edge and suppressing all resulting degree-2-vertices.)

"$\Leftarrow$": For the sufficiency part, suppose that each nontrivial component of $\overline{G \setminus M_1}$ is 3-edge-colorable with colors $\{a_1, b_1, c_1\}$, and each nontrivial component of $\overline{G \setminus M_2}$ is colored with $\{a_2, b_2, c_2\}$, and edges of the trivial components (vertexless loops, if they exist) of $\overline{G \setminus M_i}$ are colored by a_i for $i = 1, 2$. A coloring $G \setminus M_i$ is obtained by inserting those

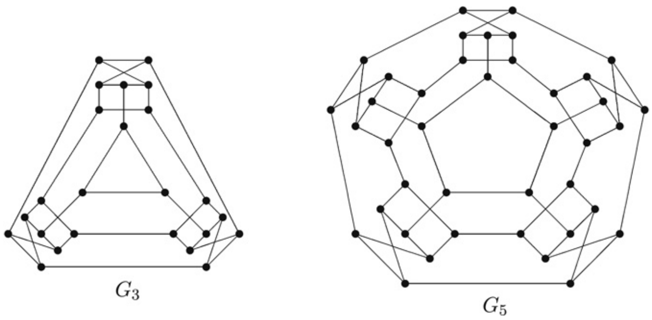

Fig. 1. Two small Goldberg snarks.

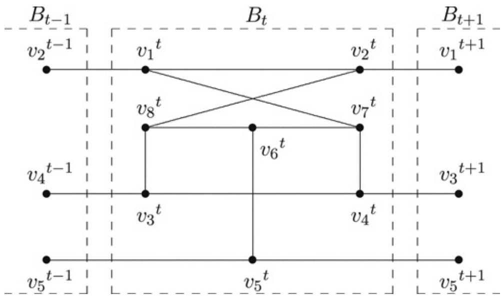

Fig. 2. The internal structure of a block and inter-block adjacency relation between blocks.

suppressed degree-2-vertices for $i = 1, 2$ and coloring each edge incident with degree-2-vertex with the same color of the edge before the vertex insertion.

A proper 6-edge-coloring of $2G$ is obtained as follows. Let e_1, e_2 be a pair of parallel edges with end-vertices u, v in $2G$.

Case 1: If there is a corresponding edge in each $G \setminus M_1$ and $G \setminus M_2$ with end-vertices u, v, then each e_i is colored the same color of the corresponding edge with end-vertices u, v in $G \setminus M_i$ for $i = 1, 2$.

Case 2: If there is only one corresponding edge with end-vertices u, v in one of $G \setminus M_1$ and $G \setminus M_2$ (not both), then e_1, e_2 are colored with colors of $\{a_i, b_i, c_i\} \setminus \{x_i\}$ where x_i is the color of the corresponding edge in $G \setminus M_i$.

This completes the proof of the lemma. ■

4. Goldberg snarks

Goldberg [10] constructed an infinite family of snarks, $G_3, G_5, G_7, \ldots$, which can be used to give infinitely many counter-examples to the critical graph conjecture [4]. Small examples G_3 and G_5 are illustrated in Fig. 1.

For every odd $k \geq 3$, the vertex set of the *Goldberg snark* G_k is: $V(G_k) = \{{v_j}^t : 1 \leq t \leq k, 1 \leq j \leq 8\}$, where the superscript t is read modulo k. The subgraph B_t induced by $\{v_1^t, v_2^t, \ldots, v_8^t\}$ is a basic block. The Goldberg snark is constructed by joining each basic block B_t with B_{t-1} and B_{t+1} (mod k). The internal adjacency relation of B_t and the inter-block adjacency relation between B_t, B_{t-1} and B_{t+1} is illustrated in Fig. 2

Theorem 4.1. *The Goldberg snark graph G_k admits a Berge–Fulkerson coloring.*

Proof. By Lemma 3.1, it is sufficient to show that G_k has a pair of edge-disjoint matchings M_1, M_2 such that $M_1 \cup M_2$ is a cycle C of G_k, and for $i = 1, 2$, each nontrivial component of $G_k \setminus M_i$ is 3-edge-colorable.

Let $C = C_1 \cup C_2$: C_1 is the circuit $v_1^1 v_2^1 v_1^2 v_2^2 \cdots v_1^k v_2^k v_1^1$ of length $2k$ and C_2, the circuit $v_3^1 v_4^1 v_3^2 v_8^2 v_6^2 v_7^2 v_4^2 \cdots v_3^k v_8^k v_6^k v_7^k v_4^k v_3^1$ of length $5k - 3$. (Note: C_2 passes through only two vertices in the block B_1, while it passes through five vertices in all other blocks G_i with $i \neq 1$.) Since k is odd, each C_i is an even length circuit of G_k (see Fig. 3)

Let M_1, M_2 be the two perfect matchings of C as follows:

$$M_1 = \{v_2^1 v_1^2, v_2^2 v_1^3, \ldots, v_2^k v_1^1\} \cup \{v_3^1 v_4^1, v_3^2 v_8^2, v_6^2 v_7^2, v_4^2 v_3^3, \ldots, v_8^k v_6^k, v_7^k v_4^k\},$$
$$M_2 = \{v_1^1 v_2^1, v_1^2 v_2^2, \ldots, v_1^k v_2^k\} \cup \{v_4^1 v_3^2, v_8^2 v_6^2, v_7^2 v_4^2, \ldots, v_3^k v_8^k, v_6^k v_7^k, v_4^k v_3^1\}.$$

Here, $C = M_1 \cup M_2$.

Note that the edges of M_1 and M_2 are selected differently in B_1 than in other B_i's ($i = 2, 3, \ldots, k$). In order to distinguish the difference, in Fig. 3, blocks $B_2, \ldots, B_k$ are lined up in the top row while B_1 is placed in the lower row, and edges of C_1 and C_2 are highlighted.

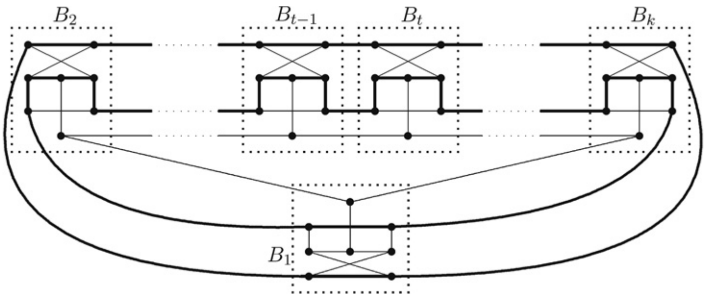

Fig. 3. $C = M_1 \cup M_2$.

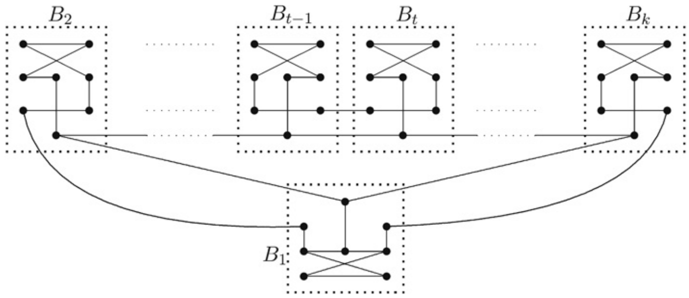

Fig. 4. $G_k \setminus M_1$.

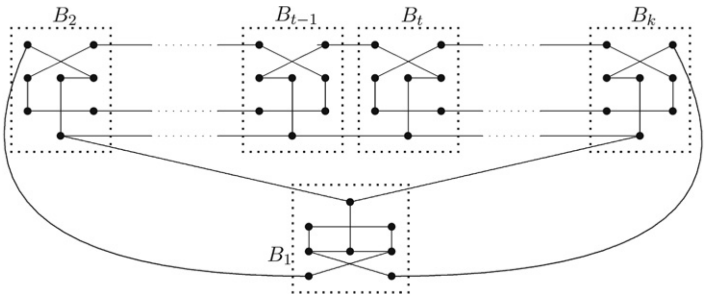

Fig. 5. $G_k \setminus M_2$.

For the suppressed graph $\overline{G_k \setminus M_1}$, there is a Hamilton circuit

$$v_5^1v_6^1v_8^1v_7^1v_5^2v_5^3\cdots v_5^k = v_5^1v_6^1v_8^1(v_2^1v_1^1)v_7^1(v_4^1v_3^2v_4^2v_7^2v_1^2v_2^2v_8^2v_6^2)v_5^2v_5^3\cdots v_5^k$$

(note that vertices listed in parenthesis are degree-2-vertices of $G_k \setminus M_1$, while all others are degree-3-vertices in both $G_k \setminus M_1$ and $\overline{G_k \setminus M_1}$). Thus $\overline{G_k \setminus M_1}$ is edge-3-colorable (see Fig. 4).

The suppressed graph $\overline{G_k \setminus M_2}$ is the union of a cubic component and $\frac{k-1}{2}$ trivial components (vertexless loops). The only cubic component has a Hamilton circuit

$$v_5^1v_6^1v_7^1v_8^1v_5^2v_5^3\cdots v_5^k = v_5^1v_6^1v_7^1(v_4^1v_3^1)v_8^1(v_2^1v_1^2v_7^2v_6^2)v_5^2v_5^3\cdots v_5^k$$

(note that vertices listed in parenthesis are degree-2-vertices of $G_k \setminus M_2$, while all others are degree-3-vertices in both $G_k \setminus M_2$ and $\overline{G_k \setminus M_2}$). Thus $\overline{G_k \setminus M_2}$ is also 3-edge-colorable (see Fig. 5). ■

5. The flower snark

Definition 5.1. For an odd integer $k \geq 3$, the flower snark J_k is constructed as following [11]: the vertex set of J_k consists of $4k$ vertices:

$$\{v_1, v_2, \ldots, v_k\} \cup \{u_1^1, u_1^2, u_1^3, u_2^1, u_2^2, u_2^3, \ldots, u_k^1, u_k^2, u_k^3\}.$$

The graph is comprised of a circuit $u_1^1u_2^1\cdots u_k^1u_1^1$ of length k and a circuit $u_1^2u_2^2\cdots u_k^2u_1^3u_2^3\cdots u_k^3u_1^2$ of length $2k$, and in addition, each vertex v_i $(i = 1, 2, \ldots, k)$ is adjacent to u_i^1, u_i^2 and u_i^3 (see Fig. 6).

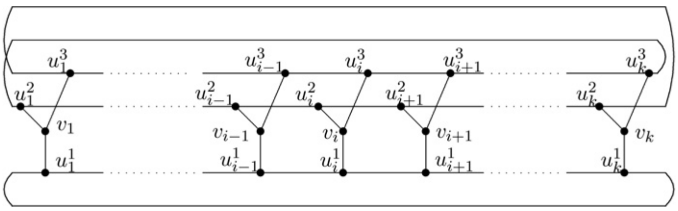

Fig. 6. Flower snarks.

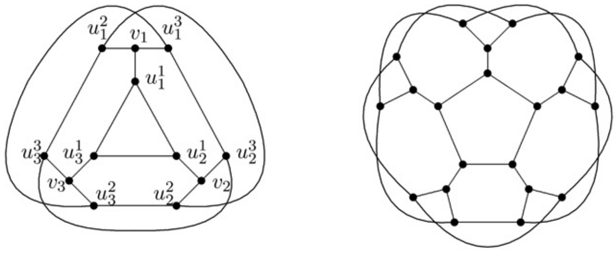

Fig. 7. Two small flower snarks.

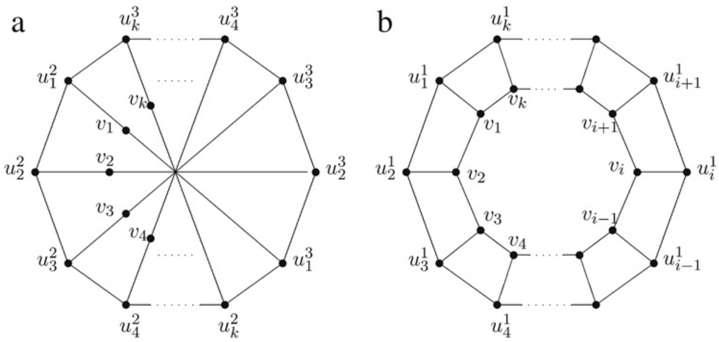

Fig. 8. (a) $\overline{H}$; (b) $\overline{G \setminus M_j}$.

The first flower snark J_3 can be obtained from the Peterson graph, with a vertex replaced by a triangle (or, equivalently, the Petersen graph can be obtained from J_3 by contracting the center triangle $u_1^1u_2^1u_3^1u_1^1$). In Fig. 7, we illustrate the first two flower snarks J_3 and J_5.

Theorem 5.2. *The flower snark graph J_k admits a Berge–Fulkerson coloring.*

Proof. By the definition of flower snark, for the odd number k, the vertex set of J_k consists of $4k$ vertices $v_1, v_2, \ldots, v_k$ and $u_1^1, u_1^2, u_1^3, u_2^1, u_2^2, u_2^3, \ldots, u_k^1, u_k^2, u_k^3$. The graph is comprised of a circuit $C' = u_1^1u_2^1 \cdots u_k^1u_1^1$ of length k and a circuit $C'' = u_1^2u_2^2 \cdots u_k^2u_1^3u_2^3 \cdots u_k^3u_1^2$ of length $2k$, and in addition, each vertex v_i ($i = 1, 2, \ldots, k$) is adjacent to u_i^1, u_i^2 and u_i^3.

By Lemma 3.1, it is sufficient to show that J_k has a pair of edge-disjoint matchings M_1, M_2 such that $M_1 \cup M_2$ is a cycle of J_k, and for each $i = 1, 2, \overline{J_k \setminus M_i}$ contains a Hamilton circuit.

Let M_1, M_2 be the two perfect matchings of C'' as follows:

$$M_1 = \{u_1^2u_2^2, u_3^2u_4^2, \ldots, u_k^2u_1^3, u_2^3u_3^3, u_4^3u_5^3, \ldots, u_{k-1}^3u_k^3\},$$
$$M_2 = \{u_2^2u_3^2, u_4^2u_5^2, \ldots, u_{k-1}^2u_k^2, u_1^3u_2^3, u_3^3u_4^3, \ldots, u_{k-2}^3u_{k-1}^3, u_k^3u_1^2\}.$$

Here $C'' = M_1 \cup M_2$.

Let H be the subgraph of J_k induced by the vertex subset $\{v_i, u_i^2, u_i^3 \mid i = 1, 2, \ldots, k\}$ (Fig. 8(a)). Here, in the suppressed graph $\overline{H}$, the circuit $C'' = u_1^2u_2^2 \cdots u_k^2u_1^3u_2^3 \cdots u_k^3u_1^2$ becomes a Hamilton circuit and $\{u_i^2v_iu_i^3 \mid i = 1, 2, \ldots, k\}$ is the set of chords of the circuit C'' (note that each v_i is a degree-2-vertex in H).

Note that matchings M_1 and M_2 form a decomposition of the circuit C''. Since k is odd, $H \setminus M_j$ is a circuit passing through the vertices $v_1, v_2, \ldots, v_k$ in this order, for each $j = 1, 2$ (see Fig. 8(a)). Hence, $G \setminus M_j$ is the graph consisting of two circuits: $H \setminus M_j$ and $C' = u_1^1u_2^1 \cdots u_k^1u_1^1$, which are joined by the set of edges $\{v_iu_i^1 | i = 1, 2, \ldots, k\}$. It is easy to see that $\overline{G \setminus M_j}$ is a planar prism, and therefore contains a Hamilton circuit and furthermore is 3-edge-colorable (see Fig. 8(b)). ■

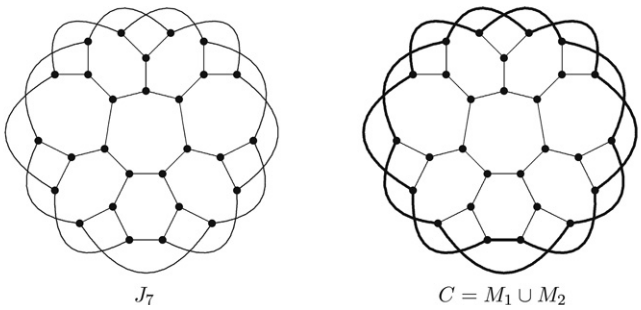

Fig. 9. Different drawings of the flower J_7.

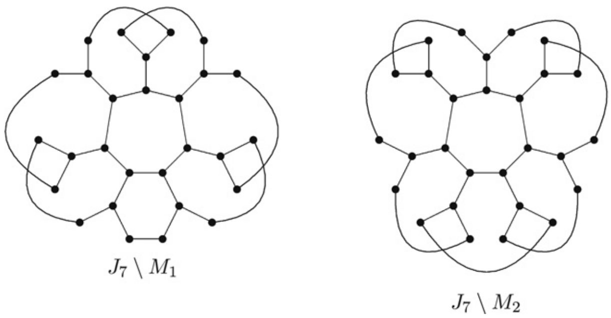

Fig. 10.

In Figs. 9 and 10, different drawings of a flower snark J_7 are presented. Together with matchings M_1 and M_2, these traditional drawings may help some readers with a different view for the structures of these matchings and Hamilton circuits in the proof of Theorem 5.2.

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