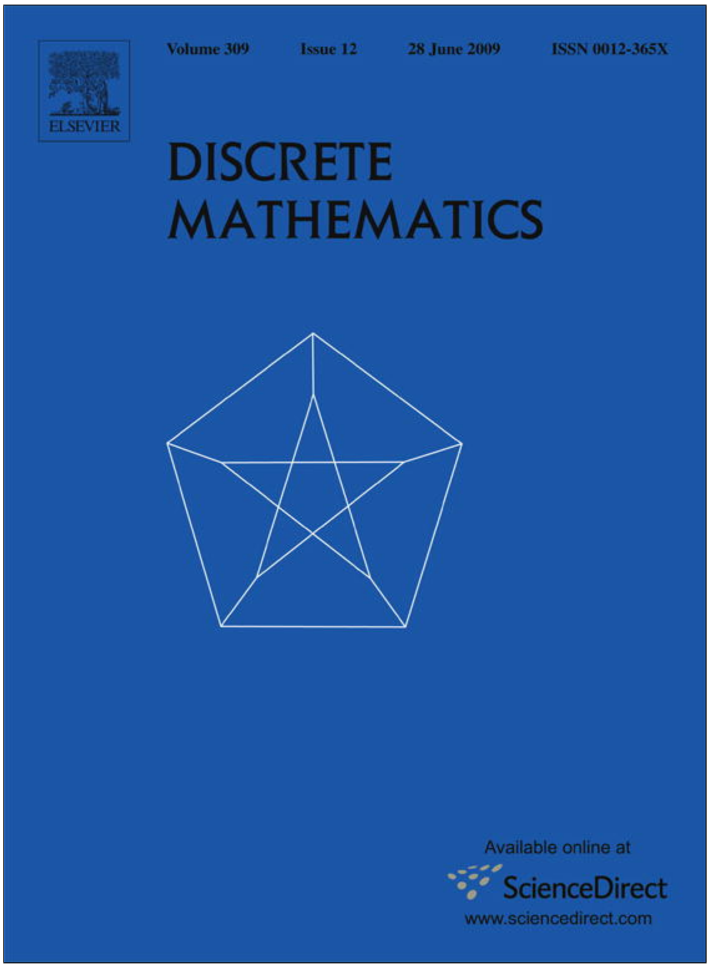
Volume 309
Issue 12
28 June 2009
ISSN 0012-365X
ELSEVIER
DISCRETE
MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 309 (2009) 4095–4107

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cliques, minors and apex graphs

Jianbing Niu, Cun-Quan Zhang *

Department of Mathematics, West Virginia University, Morgantown, West Virginia, 26506-6310, United States

ARTICLE INFO

Article history:
Received 25 May 2007
Received in revised form 1 August 2008
Accepted 8 December 2008
Available online 7 January 2009

Keywords:
Clique
Clique-minor
Apex

ABSTRACT

In this paper, we proved the following result: Let G be a $(k+2)$-connected, non-$(k-3)$-apex graph where $k \geq 2$. If G contains three k-cliques, say L_1, L_2, L_3, such that $|L_i \cap L_j| \leq k-2 (1 \leq i < j \leq 3)$, then G contains a K_{k+2} as a minor. Note that a graph G is t-*apex* if $G - X$ is planar for some subset $X \subseteq V(G)$ of order at most t.

This theorem generalizes some earlier results by Robertson, Seymour and Thomas [N. Robertson, P.D. Seymour, R. Thomas, Hadwiger conjecture for K_6-free graphs, Combinatorica 13 (1993) 279–361.], Kawarabayashi and Toft [K. Kawarabayashi, B. Toft, Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, Combinatorica 25 (2005) 327–353] and Kawarabayashi, Luo, Niu and Zhang [K. Kawarabayashi, R. Luo, J. Niu, C.-Q. Zhang, On structure of k-connected graphs without K_k-minor, Europ. J. Combinatorics 26 (2005) 293–308].

1. Introduction

Hadwiger's Conjecture from 1943 suggests a far reaching generalization of the Four Color Problem, and it is one of the most famous problems in the theory of graph minors. Hadwiger's Conjecture states the following.

Conjecture 1.1 (*Hadwiger [6]*). *For all $k \geq 1$, every k-chromatic graph has the complete graph K_k on k vertices as a minor.*

For $k = 1, 2, 3$, this conjecture is easy to prove, and for $k = 4$, Hadwiger himself [6] and Dirac [5] proved it. For $k = 5$, however, it seems extremely difficult. In 1937, Wagner [20] proved that the case $k = 5$ is equivalent to the Four Color Theorem [1,2,14]. In 1993, Robertson, Seymour and Thomas [15] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. Hence, assuming the Four Color Theorem, the case $k = 6$ of Hadwiger's Conjecture holds. This result is the deepest in this research area. So far, the cases $k \geq 7$ are open.

The following question is motivated by Hadwiger's Conjecture.

Question 1.2. *Is it true that a minimal counterexample to Hadwiger's Conjecture for $k \geq 6$ has a set X of $k-5$ vertices such that $G - X$ is planar?*

This is true for $k = 6$ as Robertson, Seymour and Thomas [15] showed. To consider a minimal counterexample to Hadwiger's Conjecture, one may try to prove the following conjecture.

Conjecture 1.3. *A minimal counterexample to Hadwiger's Conjecture is k-connected.*

This is true for $k \leq 7$ as Mader proved in [11]. Note that Toft [19] proved that a minimal counterexample to Hadwiger's Conjecture is k-edge-connected. This is a strong evidence for Conjecture 1.3.

* Corresponding author.
E-mail addresses: niu@math.wvu.edu (J. Niu), cqzhang@math.wvu.edu (C.-Q. Zhang).

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.12.009

Question 1.2 and Conjecture 1.3 lead us to the following question.

Question 1.4. *Is it true that a K_k-minor-free k-connected graph for $k \geq 6$ has a set X of $k-5$ vertices such that $G-X$ is planar?*

The case $k = 6$ is a well-known conjecture due to Jorgensen [7], and still open. If true, this would imply Hadwiger's Conjecture for the $k = 6$ case by Mader's result [12]. The case $k = 7$ was conjectured in [10] as well.

Even though the case $k = 6$ of Question 1.4 is still open, Robertson, Seymour and Thomas [15] gave a result for searching for a K_6-minor.

Theorem 1.5 (*Robertson, Seymour and Thomas [15]*). *Let G be a simple 6-connected non-apex graph. If G contains three 4-cliques, say, L_1, L_2, L_3, such that $|L_i \cap L_j| \leq 2$ $(1 \leq i < j \leq 3)$, then G contains a K_6 as a minor.*

In 2005, Kawarabayashi and Toft [10] proved the following theorem.

Theorem 1.6 (*Kawarabayashi and Toft [10]*). *Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor.*

This settles the case (6, 1) of the following conjecture known as the $(k-1, 1)$-Minor Conjecture, which is a relaxed version of Hadwiger's Conjecture.

Conjecture 1.7 (*Chartrand, Geller, Hedetniemi [3]; Woodall [21]*). *For all $k \geq 1$, every k-chromatic graph has either a K_k-minor or a $K_{\lfloor \frac{k+1}{2} \rfloor, \lceil \frac{k+1}{2} \rceil}$-minor.*

In [10], the following result is the key lemma, which gives a result for searching for a K_7-minor.

Theorem 1.8 (*Kawarabayashi and Toft [10]*). *Let G be a 7-connected graph. Suppose G contains three 5-cliques, say, L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7-minor.*

In 2005, Kawarabayashi, Luo, Niu and Zhang [9] proved the following theorem.

Theorem 1.9 (*Kawarabayashi, Luo, Niu and Zhang [9]*). *Let G be a $(k+2)$-connected graph where $k \geq 5$. If G contains three k-cliques, say L_1, L_2, L_3, such that $|L_1 \cup L_2 \cup L_3| \geq 3k-3$, then G contains a K_{k+2} as a minor.*

Our work is motivated by Theorem 1.5, and the main result of this paper is the following theorem which generalizes Theorems 1.5, 1.8 and 1.9.

Theorem 1.10. *Let G be a $(k+2)$-connected, non-$(k-3)$-apex graph where $k \geq 2$. If G contains three k-cliques, say L_1, L_2, L_3, such that $|L_i \cap L_j| \leq k-2 (1 \leq i < j \leq 3)$, then G contains a K_{k+2} as a minor.*

Theorem 1.10, which generalizes Theorems 1.5, 1.8 and 1.9, implies that a $(k+2)$-connected K_{k+2}-minor-free graph cannot contain three *"nearly" disjoint* k-cliques.

A remark about the extreme case in Theorem 1.10: $(k-3)$-apex graph: a $(k+2)$-connected graph may contain many copies of k-clique, but not necessarily a K_{k+2}-minor. For example, the graph $G = K_{k-3} + G_1$, where G_1 is a 5-connected planar graph, is obviously K_{k+2}-minor-free and contains many copies of k-clique, many pairs of which overlap with each other with only $(k-3)$ vertices (in K_{k-3}).

We hope our result could be used to prove some results on 7- and 8-chromatic graphs. In fact, in [8], Kawarabayashi proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor by applying Theorem 1.9. We expect that Theorem 1.10 would be useful in the proofs of some h-chromatic cases of Conjecture 1.1 or Conjecture 1.7 for some larger integers h. Note that Theorem 1.9 would imply the 7-chromatic case of Hadwiger's Conjecture (Conjecture 1.1) if one could find three copies of 5-clique not to overlap too much with each other, since Mader proved that the connectivity of such a counterexample is at least 7 [11].

The following was conjectured by Seymour and Thomas.

Conjecture 1.11. *For every $p \geq 1$, there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \frac{(p-1)(p-2)}{2} + 1$ edges has a K_p-minor.*

Note that the connectivity condition and the condition of the order of graphs are necessary because random graphs having no K_k-minor may have average degree $k\sqrt{\log k}$, but all these graphs are small. So if a graph is large enough and highly connected, we do not know any construction of infinite family of counterexamples. This conjecture is true for $p \leq 9$. For $p \leq 7$, these conjecture was proved by Mader [11]. For $p = 8$, Jorgensen [7] proved it. Very recently, Song and Thomas [17] proved the case $p = 9$. Note that all of these results do not require the connectivity condition in this conjecture.

2. Terminology and notations

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. The complete graph (or, clique, as a subgraph) on n vertices is denoted by K_n and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices is denoted by $K_{n,m}$.

A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and vertices and contracting edges.

For a vertex x of a subgraph H_1 of G, the neighborhood of x in H_1 is denoted by $N_{H_1}(x)$. And, for a vertex $v \in V(G)$ and a vertex subset (or a subgraph) Y of G, $d_Y(x) = |\{v \in Y : xv \in E(G)\}|$. A graph G is *k-chromatic* if G is vertex-k-colorable but not vertex-$(k-1)$-colorable. Let V_1 and V_2 be subsets of $V(G)$. The symmetric difference of V_1 and V_2, denoted by $V_1 \Delta V_2$, is the set $(V_1 \cup V_2) - (V_1 \cap V_2)$.

Let us say a graph G is *k-apex* if $G - X$ is planar for some subset $X \subseteq V(G)$ with $|X| \leq k$. By the definition, if $k \leq 0$, then a *k-apex* is planar. (For technical reason, a k-apex with negative k is mentioned sometime in this paper. Note that, there is no subset X with negative order. Hence, a k-apex with $k < 0$ is actually a planar graph: since G is already planar after the deletion of a subset X that does not exit.) Furthermore, (a) for $k \geq 1$, a graph G is non-*k-apex* if $G - X$ is not planar for every subset $X \subseteq V(G)$ with $|X| \leq k$; (b) for $k = 0$, a graph G is non-*k-apex* if G itself is not planar; (c) for $k < 0$, a non-*k-apex* graph is either planar or non-planar. (Similar as above, for a graph G to be a non-*k-apex* with $k < 0$, it is necessary that there is a subset X of order at least 0 such that $G - X$ is planar.)

A subset $X \subseteq V(G)$ is a *fragment* of G if $X \neq \emptyset$ and X induces a connected subgraph of G. Subsets $X, Y \subseteq V(G)$ are *adjacent* in G if $X \cap Y = \emptyset$ and some $x \in X$ is adjacent in G to some $y \in Y$.

A *cluster* in G is a set of mutually adjacent fragments G, and it is a *p-cluster* if it has cardinality p. Thus G has a K_p-minor if and only if it has a p-cluster. Given a subset $Y \subseteq V(G)$, a *p-cluster* $\wp$ is said to *traverse* Y if $\wp = \{X_1, X_2, \ldots, X_p\}$ in such a way that $X_i \cap Y \neq \emptyset \ (1 \leq i \leq p)$.

Let v_1, v_2, v_3 be mutually adjacent vertices of a graph G. We say G is *triangular* with respect to v_1, v_2, v_3 if G is simple and either

(i) for some $i \ (1 \leq i \leq 3)$, $G - v_i$ has maximum degree at most 2, and either $G - v_i$ is a cycle or it has no cycle, or

(ii) all vertices of G have degree at most 3, there is at most one vertex v of degree 3 with $v \neq v_1, v_2, v_3$, and $G - v_1 - v_2 - v_3$ has no cycle, or

(iii) all vertices of G have degree at most 3, there is a triangle C in $G - v_1 - v_2 - v_3$, every vertex of degree 3 is in $\{v_1, v_2, v_3\} \cup V(C)$, and every cycle except for the two triangles $\{v_1, v_2, v_3\}$ and C contains both a vertex in $\{v_1, v_2, v_3\}$ and $V(C)$.

3. Lemmas

3.1. Good paths

One of the key lemmas in our proof is Mader's "H-Wege" Theorem, which was proved in [13].

Lemma 3.1 (*Mader [13]*). *Let G be a graph, let $S \subseteq V(G)$ be an independent set, and $k \geq 0$ be an integer. Then exactly one of the following two statements holds.*

(1) *There are k paths of G, each with two distinct ends both in S, such that each $v \in V(G) - S$ is in at most one of the paths.*
(2) *There exists a vertex set $W \subseteq V(G) - S$ and a partition $Y_1, \ldots, Y_n$ of $V(G) - (S \cup W)$, and a subset $X_i \subseteq Y_i$, $1 \leq i \leq n$, such that*
   (a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor < k$,
   (b) *no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and,*
   (c) *every path of $G - W$ with distinct ends both in S has an edge with both ends in Y_i for some i.*

Let $Z_1, Z_2, \ldots, Z_h$ be subsets of $V(G)$. A path P of G with ends u, v is said to be *good* if there exist distinct i, j with $1 \leq i, j \leq h$ such that $u \in Z_i$ and $v \in Z_j$.

As Robertson, Seymour and Thomas pointed out in [15], we can deduce the following lemma from Lemma 3.1.

Lemma 3.2 (*Robertson, Seymour and Thomas [15]*). *Let G be a graph, let $Z_1, Z_2, \ldots, Z_h$ be subsets of $V(G)$, and let $k \geq 1$ be an integer. Then exactly one of the following two statements holds.*

(1) *There are k mutually disjoint good paths of G.*
(2) *There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \ldots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that*
   (a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor < k$,
   (b) *for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^{h} Z_j) \subseteq X_i$, and*
   (c) *every good path P in $G - W$ has an edge with both ends in Y_i for some i.*

3.2. Cluster

Lemma 3.3 (*Robertson, Seymour and Thomas [15], page 291*). *Let v_1, v_2, v_3 be mutually adjacent vertices of a 4-connected simple non-planar graph G. Let $\Im \subseteq V(G)$ with $v_1, v_2, v_3 \in \Im$ such that $\Im$ is not triangular. Then there is a 5-cluster $\wp = \{\{v_1\}, \{v_2\}, \{v_3\}, X_1, X_2\}$ in G such that $\wp$ traverses $\Im$.*

The following lemma is an immediate corollary of a result by Robertson, Seymour and Thomas in [15] (page 288).

Lemma 3.4. *Let G be a 4-connected graph and $\Im \subseteq V(G)$ with $|\Im| = 4$. Then either*

(i) *there is a 4-cluster in G traversing $\Im$, or*
(ii) *G can be drawn in a plane so that every vertex in $\Im$ is incident with the infinite region.*

3.3. The 6-cluster lemma

The following lemma deals with an extreme case of our main theorem. Since the proof of the lemma is relatively long and complicated, we present it here as an independent lemma and its proof in Section 5. Readers may postpone the reading of the proof of Lemma 3.5 until after the proof of the main theorem.

Lemma 3.5. *Let x_i, y_i, z_i $(1 \le i \le 3)$ be distinct vertices of a 6-connected simple graph G, such that $\{x_1, y_1, z_2, z_3\}$, $\{x_2, y_2, z_3, z_1\}$, $\{x_3, y_3, z_1, z_2\}$ are 4-cliques. Suppose, that there is a partition Y_1, Y_2 of $V(G) - \{z_1, z_2, z_3\}$ with $x_1, x_2, x_3 \in Y_1$, and $y_1, y_2, y_3 \in Y_2$, such that x_iy_i $(1 \le i \le 3)$ are the only edges of G with one end in Y_1 and the other in Y_2. Then G has a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$.*

Note that, Robertson, Seymour and Thomas gave a result in page 293 of [15] similar to Lemma 3.5. However, in order to obtain a sharper and more general result in our main theorems (Theorems 1.10 and 4.1), we need a stronger result in Lemma 3.5 (for 6-cluster instead of 6-minor), which is approached differently from that in [15].

4. Proof of the main theorem

The main theorem (Theorem 1.10) is to be proved in this section. Here we prove a theorem that is slightly stronger than the main theorem (Theorem 1.10).

Theorem 4.1. *Let G be a $(k+2)$-connected, non-$(k-3)$-apex graph where $k \ge 2$. If G contains three k-cliques, say L_1, L_2, L_3, such that $|L_i \cap L_j| \le k-2 (1 \le i < j \le 3)$, then one of the following holds,*

(1) *G contains a $(k+2)$-cluster traversing $L_1 \cup L_2 \cup L_3$, or*
(2) *(an exceptional case) $|T| = k-2$ where $T = L_1 \cap L_2 \cap L_3$, and $G - T$ is a planar graph with all edges in $L_i - T$ $(i = 1, 2, 3)$ around the exterior face. In this case, G contains a $(k+2)$-cluster $\{\{v_1\}, \ldots, \{v_k\}, B, I\}$ where $L_1 = \{v_1, \ldots, v_k\}$, B is the set of all vertices of $G - T$ around the exterior face except for those in L_1, and I is the set of all interior vertices of $G - T$.*

Note: readers might be confused by a non-$(k-3)$-apex graph if $k = 2$. Recall that a graph H is a non-t-apex if $G - R$ is planar for some vertex subset R, then R must be of order at least $t+1$. Hence, a non-$(k-3)$-apex graph for $k = 2$ can be any graph, planar or non-planar.

Proof. Let G be a counterexample to the theorem with k as small as possible.

4.1. We claim that $k \ge 3$

For otherwise, we may assume $k = 2$, G is 4-connected graph, and G contains three disjoint 2-cliques, say L_1, L_2, L_3.
Since L_1 and L_2 are disjoint 2-cliques, $|L_1 \cup L_2| = 4$. Note that G is 4-connected, by Lemma 3.4. There are two cases:
(1) There is a 4-cluster in G traversing $L_1 \cup L_2$. In this case, by the definition of cluster, this 4-cluster in G also traverses $L_1 \cup L_2 \cup L_3$, a contradiction, hence we are done.
(2) G can be drawn in a plane so that every vertex in $L_1 \cup L_2$ is incident with the infinite region. In this case, since G is 4-connected, the edges of L_1 and L_2 must be around the exterior face. If one vertex v_1 of L_3 is not incident with the infinite region, then there are four internal vertex-disjoint paths from v_1 to $L_1 \cup L_2$, hence we get a 4-cluster traversing $L_1 \cup L_2 \cup L_3$, a contradiction. Therefore two vertices of L_3 must be incident with the infinite region. Note that G is 4-connected, G has a 4-cluster $\{\{v_1\}, \{v_2\}, B, I\}$ where $L_1 = \{v_1, v_2\}$, B is the set of all vertices of G around the exterior face except for those in L_1, and I is the set of all interior vertices of G (it is easy to see that I and B both are connected), a contradiction. ∎

4.2. We claim that $|L_1 \cap L_2 \cap L_3| = 0$

For otherwise, we assume $|L_1 \cap L_2 \cap L_3| \ne 0$. Let $x \in L_1 \cap L_2 \cap L_3$ and $G' = G - \{x\}$, then G' is a $(k+1)$-connected non-$(k-4)$-apex graph. By minimality of k, there are two cases:

Case (1): There is a $(k+1)$-cluster $\wp_1$ of $G - \{x\}$ traversing $L_1 \cup L_2 \cup L_3 - \{x\}$. Let $\wp = \wp_1 \cup \{\{x\}\}$, then $\wp$ is a $(k+2)$-cluster traversing $L_1 \cup L_2 \cup L_3$, a contradiction.

Case (2): $|T'| = k-3$ where $T' = (L_1 \cap L_2 \cap L_3) - \{x\}$, and $G' - T'$ is a planar graph with all edges of $(L_i - \{x\}) - T'$ around the exterior face. G' contains a $(k+1)$-cluster $\wp' = \{\{v_1\}, \ldots, \{v_{k-1}\}, B, I\}$ where $\{x, v_1, \ldots, v_{k-1}\} = L_1$ and B is the set of all vertices of $G' - T'$ around the exterior face except for those in L_1, and I is the set of all interior vertices of $G' - T'$.

In order to show that $\wp = \{\{x\}, \{v_1\}, \ldots, \{v_{k-1}\}, B, I\}$ is a $(k+2)$-cluster of G, it is sufficient to prove that x is adjacent to every other fragment. (i) it is obvious that $v_i \in N(x)$ since $x \in L_1$; (ii) it is similar that $N(x) \cap B \neq \emptyset$ since $x \in L_2$ and $L_2 \cap B \neq \emptyset$; (iii) $N(x) \cap I \neq \emptyset$ for otherwise, the vertex x can be embedded into the exterior face of $G' - T'$ and therefore, $G - T'$ is planar. This contradicts that G is non-$(k-3)$-apex.

4.3. We claim that $k \geq 4$

For otherwise, by (4.1), we may assume $k = 3$. That is, G is a 5-connected non-planar graph, and G contains three 3-cliques, say L_1, L_2, L_3, such that $|L_i \cap L_j| \leq 1 (1 \leq i < j \leq 3)$. By (4.2), we have $|L_1 \cap L_2 \cap L_3| = 0$.

Let $Z = L_1 \cup L_2 \cup L_3$ and $v_1, v_2, v_3 \in L_1$. Then Z is not triangular with respect to v_1, v_2, v_3. By Lemma 3.3, there is a 5-cluster $\wp$ in G such that $\wp$ traverses Z. ■

4.4. We claim that $|L_i \cap L_j| \leq 1$ for $1 \leq i < j \leq 3$

For otherwise, we may assume $|L_1 \cap L_2| \geq 2$. Let $B \subseteq L_1 \cap L_2$ with $|B| = 2$. By (4.2), $B \cap L_3 = \emptyset$ since $L_1 \cap L_2 \cap L_3 = \emptyset$. Since $G - B$ is k-connected, there exist k disjoint paths from L_3 to $L_1 \cup L_2 - B$. Let $x, y \in B$ and $P_1, P_2, \ldots, P_k$ be a set of disjoint paths from L_3 to $L_1 \cup L_2 - B$. Then $\wp = \{P_1, P_2, \ldots, P_k, x, y\}$ is a $(k+2)$-*cluster* that traverses $L_1 \cup L_2 \cup L_3$, a contradiction. ■

4.5. A path P of G with ends u, v is said to be good if there exist distinct i, j with $1 \leq i, j \leq 3$ such that $u \in L_i$ and $v \in L_j$

4.6. We claim that there do not exist $(k+2)$ mutually disjoint good paths in G

Let $P_1, P_2, \ldots, P_{k+2}$ be a set of disjoint *good* paths of G. Then $\wp = \{P_1, P_2, \ldots, P_{k+2}\}$ is a $(k+2)$-*cluster* that traverse $L_1 \cup L_2 \cup L_3$. ■

By Lemma 3.2 and (4.6), we have the following structure of G:

4.7. There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \ldots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that

(a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor \leq k+1$,
(b) for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^{3} L_j) \subseteq X_i$, and
(c) every good path P in $G - W$ has an edge with both ends in Y_i for some i.

Let $M = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$, and choose W and $Y_1, X_1, \ldots, Y_n, X_n$ such that $|W|$ **is as large as possible**. Without loss of generality, we can assume that $Y_i \neq \emptyset$ for any $i \in \{1, 2, \ldots, n\}$. By the definition of W, M and (4.7)(c), we have the following immediate observations:

4.8

(a) $M \subseteq W$ by (4.7)(c).
(b) $|L_1 \cup L_2 \cup L_3| = |L_1| + |L_2| + |L_3| - |M|$ by definition of M and (4.2).
(c) $|M| \leq 3$ by (4.2 and 4.4).
(d) $|L_i \cup L_j| > k+2$ for $1 \leq i < j \leq 3$.
(4.8)(d) is proved as follows: by (4.4) and $k \geq 4$ (by (4.3))

$$|L_i \cup L_j| = |L_i| + |L_j| - |L_i \cap L_j| = 2k - 1 > k + 2.$$

The following claim (e) follows from assumption (4.7)(b).
(e) $W \cup X_1 \cup \cdots \cup X_n \supseteq L_1 \cup L_2 \cup L_3$, and $|W| + \sum_{i=1}^{n} |X_i| \geq |L_1 \cup L_2 \cup L_3|$.

4.9. We claim that $X_i \neq \emptyset$ for all i

Suppose that $X_i = \emptyset$ for some i. Since $|W| \leq k+1$ (by (4.7)(a)) and $|L_1 \cup L_2 \cup L_3| \geq |L_1 \cup L_2| \geq k+2$ (by (4.8)(d), there is an integer j ($j \neq i$) such that $X_j \neq \emptyset$ (by (4.8)(e)). Hence $n \geq 2$. Since Y_i is not empty, W is a cutset that separates Y_i and non-empty X_j and is of cardinality at most $k+1$. This contradicts that G is $(k+2)$-connected.

4.10. We claim that $|X_i|$ is odd for all i

Suppose that $|X_1|$ is even, then by (4.9), $|X_1| \geq 2$. Let $v \in X_1$, $W^* = W \cup \{v\}$, $Y_1^* = Y_1 - v$, $X_1^* = X_1 - v$ and $X_i^* = X_i$, $Y_i^* = Y_i$ for $2 \leq i \leq n$. The partition $\{W^*, X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*\}$ of $V(G)$ satisfies (4.7)(a)–(c), contradicting the choice that $|W|$ is as large as possible.

4.11. We claim that

(a) $n \geq k-2$
(b) **if** $n = k-2$ **then**

$$|W| = |M| \quad \text{and} \quad W \cup X_1 \cup X_2 \cup \cdots \cup X_n = L_1 \cup L_2 \cup L_3.$$

Since $|L_1 \cup L_2 \cup L_3| \geq |L_1 \cup L_2| \geq k+2$ (by (4.8)(d)) and $|W| \leq k+1$ (by (4.7)(a)), we have $n \geq 1$. By (4.7)(a), (4.8)(a) and (4.8)(b), we have

$$\begin{aligned} 2(k+1) &\geq 2\left(|W| + \sum_{1\leq i\leq n} \lfloor \frac{1}{2}|X_i| \rfloor\right) = 2|W| + \sum_{1\leq i\leq n} |X_i| - n \\ &\geq |W| + |L_1 \cup L_2 \cup L_3| - n \geq |M| + |L_1 \cup L_2 \cup L_3| - n \\ &= |L_1| + |L_2| + |L_3| - n = 3k - n. \end{aligned}$$

Thus,

$$n \geq k-2$$

and if $n = k-2$, then all equalities hold and therefore,

$$|W| = |M| \quad \text{and} \quad |W| + \sum_{1\leq i\leq n} |X_i| = |L_1 \cup L_2 \cup L_3|.$$

4.12. Definition of A_i (for $i = 1, 2, 3$)

Let G'' be the subgraph obtained from $G - W$ by deleting all edges contained in any Y_j. Let A_i be the union of the vertex subsets of all components of G'' containing some vertex of L_i for each $i \in \{1, 2, 3\}$.

4.13. Properties of $\{A_1, A_2, A_3\}$

Properties of $\{A_1, A_2, A_3\}$ are to be studied in this subsection. The first property is immediate by (4.7) and the definition of A_i.

(a) $L_i - W \subseteq A_i \subseteq V(G) - W$ for $i = 1, 2, 3$.
Note that each $Y_j - X_j$ is an independent set of G'', and by (4.7)(b), we have the following properties.
(b) $A_i \subseteq X_1 \cup \cdots \cup X_n$ for $i = 1, 2, 3$.
(c) A_1, A_2, A_3 **are disjoint** by the definition of A_i and (4.7)(c).
(d) **Every path of** $G - W$ **from** A_i **to** A_{i^*} (**for** $1 \leq i < i^* \leq 3$) **has at least two vertices in** X_j **for some** j.
Proof of (d). Suppose there exists a path P from $v \in A_1$ to $u \in A_2$ in $G - W$. By the definition of A_1, A_2, we can take two disjoint paths Q and R such that Q is a path from some vertex $x \in L_1$ to v in $G[A_1]$ and R is a path from some vertex $y \in L_2$ to u in $G[A_2]$. Both Q and R have no edges with both ends in Y_j for any j by definition of A_i. Then we have a path S from x to y by using Q, P, R. Since S is a good path by (4.7)(c), S has an edge $e = x_1y_1 \in Y_j$ for some j. Note that $e \notin E(Q)$ and $e \notin E(R)$. This implies $e \in E(P)$ and $x_1, y_1 \in V(P)$. Note that, by (4.13)(b), both v and u belong to $X_1 \cup \cdots \cup X_n$. By (4.7)(b), the part of P from v to x_1 must contain a vertex from X_j, and likewise the part of P from y_1 to u. ■
(e) $|A_i| \leq k + 1 - |W|$ for $1 \leq i \leq 3$.
Proof of (e). Suppose $|A_1| \geq k + 2 - |W|$. It is obvious that $|W| \leq k + 1$ (by (4.7)(a)). Hence, $A_1 \neq \emptyset$. We also have that $L_2 \cup L_3 - W \neq \emptyset$ since $|L_2 \cup L_3| \geq k + 2$ (by (4.8)(d)) and $|W| \leq k + 1$ (by (4.7)(a)).
Since $|L_2 \cup L_3| \geq k + 2$ (by (4.8)(d)), we have that $|L_2 \cup L_3 - W| \geq k + 2 - |W|$. Note that $G - W$ is $(k + 2 - |W|)$-connected, there are $(k + 2 - |W|)$ disjoint paths from A_1 to $L_2 \cup L_3 - W$ each of which is of order at least $k + 2 - |W|$. By (4.13)(d), every path P_j contains at least two vertices of X_i for some i. Hence, $\sum_{1\leq i\leq n} \lfloor \frac{1}{2}|X_i| \rfloor \geq k + 2 - |W|$. This is a contradiction to (4.7)(a). The other cases follow by the similar arguments. ■

4.14. We claim that $|W| \leq 3$. If equality holds then $M = W$ and $|A_i| = k + 1 - |W|$

This claim is to be proved in two steps in this subsection. First we show that

(a) $\sum_{i=1}^{3} |L_i \cap W| \leq |W| + 3$.
Note that $\sum_{i=1}^{3} |L_i \cap W| \leq |W| + |M|$. Hence, $\sum_{i=1}^{3} |L_i \cap W| \leq |W| + 3$ since $|M| \leq 3$ by (4.8)(c). ■
(b) By (4.13)(a), (4.13)(e) and (4.14)(a), we have the following inequality:

$$3k = \sum_{i=1}^{3} |L_i| \leq \sum_{i=1}^{3} (|A_i| + |L_i \cap W|) \leq 3(k + 1 - |W|) + |W| + 3 = 3k + 6 - 2|W|.$$

Hence, $|W| \leq 3$. And if $|W| = 3$ then $M = W$ and $|A_i| = k + 1 - |W|$. ■

4.15. We claim that, for $1 \le j \le n$, if $|W \cup X_j| < (k+2)$ then $X_j = Y_j$

Suppose that $X_j \neq Y_j$. First we claim that $V(G) - Y_j - W$ is not empty. Since $|W \cup X_j| \le k+2$ and $|L_1 \cup L_2 \cup L_3| = 3k - |M| \ge 3k - 3 \ge k + 3$ (by (4.8)(b) and (4.8)(c)), $(L_1 \cup L_2 \cup L_3) - (W \cup X_j) \neq \emptyset$. Hence $V(G) - Y_j - W$ which contains $(L_1 \cup L_2 \cup L_3) - (W \cup X_j)$ is not empty.

Note that G is $(k+2)$-connected and by (4.7)(b), $W \cup X_j$ is a vertex-cut separating $Y_j - X_j$ and $V(G) - Y_j - W$ neither of which is empty. It follows that $|W \cup X_j| \ge (k+2)$, as required. ■

4.16. We claim that, for $1 \le j \le n$, if $|X_j| < 3$ then $X_j = Y_j$

By (4.15), it is obvious that $X_j = Y_j$ if $|X_j| < 3$ since $k \ge 4$ (by (4.3)) and $|W| \le 3$ (by (4.14)).

4.17. Let $Z = (X_1 \cup \cdots \cup X_n) - (L_1 \cup L_2 \cup L_3)$

4.18. Some vertex-cuts of G

Suppose that $X_i \cap L_j \neq \emptyset$ for some $i \in \{1, 2, \ldots, n\}, j \in \{1, 2, 3\}$. By (4.7)(c), (4.13)(a) and (4.13)(d), any path joining $X_i \cap L_j$ and $L_1 \cup L_2 \cup L_3 - W - L_j$ must use a vertex of W or Z or $X_i \Delta L_j$. Therefore, $(X_i \Delta L_j) \cup W \cup Z$ **is a cutset of** G **separating** $X_i \cap L_j$ **from** $L_1 \cup L_2 \cup L_3 - W - L_j$.

4.19. We claim that $|X_i| \ge 3$ for $1 \le i \le n$

This claim is to be proved in several steps in this subsection.

(a) First we show that, for $1 \le i \le 3$, $1 \le j \le n$, if $|X_j| = 1$, then $A_i \cap X_j = \emptyset$.
 Suppose $A_1 \cap X_j \neq \emptyset$. Let $X_j = \{v\}$ and $N = N_G(v)$. Since G is $(k+2)$-connected, $|N| \ge k+2$. Hence $|N - W| \ge k + 2 - |W|$. Note that $|A_1| \le k + 1 - |W|$ by (4.13)(e), this implies $N - A_1 - W \neq \emptyset$. Take a vertex $x \in N - A_1 - W$. Since $|X_j| = 1$, we have $X_j = Y_j = \{v\}$ by (4.16). Note that $xv \in E(G)$, x is in A_1 by the definition of A_1, a contradiction. Hence $A_1 \cap X_j = \emptyset$. ■

(b) Second we show that, for $1 \le i \le 3$, $1 \le j \le n$, if $|X_j| = 1$, then $A_i \cap N_G(X_j) = \emptyset$.
 Suppose that $|X_1| = 1$ and $x \in A_1 \cap N_G(X_1)$. Hence, by (4.13)(b), $x \in X_i$ for some $i \neq 1$. Since $|X_1| = 1$, by the definition of A_1 (defined in (4.12)), $X_1 \subseteq A_1$. This contradicts (4.19)(a) since $|X_1| = 1$. ■

(c) Since $|X_i|$ is odd for each i (by (4.10)), let m be an integer such that $m \le n$ with $|X_i| = 1$ for $1 \le i \le m \le n$ and $|X_j| \ge 3$ for $m < j \le n$.

By the definition of A_i and (4.8), we have

$$\sum_{i=1}^{3} |A_i| \ge |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| - |W| \tag{1}$$

Also, by (4.7)(a),

$$\sum_{m<j\le n} |X_j| \le 3 \sum_{m<j\le n} \lfloor \frac{1}{2}|X_j| \rfloor \le 3 \sum_{1\le j\le n} \lfloor \frac{1}{2}|X_j| \rfloor \le 3(k+1-|W|) \tag{2}$$

Assume $X = X_1 \cup X_2 \cup \cdots \cup X_m$ and $N = N_G(X) - X$. Then we can get the following.

(i) $N \subseteq W \cup X_{m+1} \cup \cdots \cup X_n$ by (4.7)(b) and (4.16).
(ii) $N \cap A_1 = N \cap A_2 = N \cap A_3 = \emptyset$ by (4.19)(b).
(iii) $|N| \ge k+2$ since N separates X from $A_1 \cup A_2 \cup A_3$ (by (4.19)(a) and (4.19)(b)) and G is (k+2)-connected.

Hence, we have

$$|N| + |A_1| + |A_2| + |A_3| \le |W| + \sum_{i=m+1}^{n} |X_i|. \tag{3}$$

By (iii), (1)–(3), we have

$$\begin{aligned}(k+2) + (3k - |M| - |W|) &\le |W| + 3(k+1-|W|)\\ &= 3k + 3 - 2|W|.\end{aligned}$$

Hence,

$$|W| \le 1 + |M| - k.$$

By (4.8)(a),

$$|W| \leq 1 + |W| - k.$$

That is,

$$k \leq 1.$$

This contradicts $k \geq 4$ (4.3) and completes the proof of (4.19). ■

4.20. We prove some inequalities for $|Z|$

(i)

$$|Z| \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ for every $j \in \{1, 2, \ldots, n\}$.

(ii)

$$|Z| \leq 3 + |M| - 2|W|,$$

and the equality holds if and only if $|X_j| = 3$ for every $j \in \{1, 2, \ldots, n\}$ and $W \subseteq L_1 \cup L_2 \cup L_3$.

Let $s = |Z|$. Then, by (4.17),

$$|X_1 \cup \cdots \cup X_n| = s + |L_1 \cup L_2 \cup L_3 - W|.$$

But, by (4.19), $|X_j| \leq 3\lfloor \frac{1}{2}|X_j| \rfloor$ for $1 \leq j \leq n$, and therefore

$$3 \sum_{1 \leq j \leq n} \lfloor \frac{1}{2}|X_j| \rfloor \geq \sum_{1 \leq j \leq n} |X_j| = s + |L_1 \cup L_2 \cup L_3 - W|,$$

with equality if and only if $|X_j| = 3$ for any $j \in \{1, 2, \ldots, n\}$. By (4.7)(a), we have

$$3(k + 1 - |W|) \geq s + |L_1 \cup L_2 \cup L_3 - W|.$$

That is,

$$s \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ for any $j \in \{1, 2, \ldots, n\}$. That completes the proof of (4.20)(i).

Note that, by (4.8)(b), we have

$$|L_1 \cup L_2 \cup L_3 - W| \geq |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| - |W|,$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$. Hence, by (4.20)(i),

$$\begin{aligned} s &\leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W| \leq 3k + 3 - 3|W| - (3k - |M| - |W|) \\ &= 3 + |M| - 2|W|, \end{aligned}$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$ and $|X_j| = 3$ for every $j \in \{1, 2, \ldots, n\}$. This completes the proof of (4.20)(ii). ■

4.21. (i) $|A_i \cap X_j| < \frac{1}{2}|X_j|$ for $1 \leq j \leq n$ and $1 \leq i \leq 3$

Suppose that $|A_1 \cap X_1| \geq \frac{1}{2}|X_1|$. Since $|X_1| \geq 3$ by (4.19), there exists a vertex $v \in A_1 \cap X_1$. Since $|L_2 \cup L_3 - W| \geq |L_2 \cup L_3| - |W| \geq k + 2 - |W|$ by (4.8)(d), and $G - W$ is $(k + 2 - |W|)$-connected, there are $(k + 2 - |W|)$ paths of $G - W$ between A_1 and $L_2 \cup L_3 - W$, disjoint except possibly for v. Choose them with no internal vertex in A_1. By (4.13)(d), each has at least two vertices in X_j for some j, but at most $\lfloor \frac{1}{2}|X_j| \rfloor$ of them have two vertices in X_j for each $j \neq 1$. Note that by (4.7)(a), we have

$$\sum_{2 \leq j \leq n} \left\lfloor \frac{1}{2}|X_j| \right\rfloor \leq k + 1 - |W| - \left\lfloor \frac{1}{2}|X_1| \right\rfloor.$$

Thus, at least $1 + \lfloor \frac{1}{2}|X_1| \rfloor$ of them have two vertices in X_1. But each has only one vertex in A_1, and so has a vertex in X_1 which does not belong to A_1, and all these vertices in $X_1 - A_1$ are different. Hence $|X_1 - A_1| \geq 1 + \lfloor \frac{1}{2}|X_1| \rfloor$, a contradiction. ■

(ii) $|L_i \cap X_j| < \frac{1}{2}|X_j|$ for $1 \leq j \leq n$ and $1 \leq i \leq 3$ by (4.13)(a) and (4.21)(i).

4.22. We eliminate the case of $k = 4$ and $|W| = 3$

In this case, by (4.11)(a), $n \geq k - 2 = 2$.
Moreover, by (4.7)(a), we have

$$|W| + \sum_{1\leq i\leq n} \lfloor \frac{1}{2}|X_i| \rfloor \leq k + 1 = 5.$$

That is,

$$\sum_{1\leq i\leq n} \lfloor \frac{1}{2}|X_i| \rfloor \leq 2.$$

We have $n = 2$ and $|X_1| = |X_2| = 3$ (by (4.19)). Hence, $Z = \emptyset$ by (4.11)(b).

Next we claim that $|L_i \cap X_j| \neq 0$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. For otherwise, we may assume that $L_1 \cap X_1 = \emptyset$. Then by (4.21)(ii), $|L_2 \cap X_1| \leq 1$ and $|L_3 \cap X_1| \leq 1$. This implies that $Z \supseteq X_1 - (L_1 \cup L_2 \cup L_3) = X_1 - L_2 \cup L_3 \neq \emptyset$. This contradicts that $Z = \emptyset$.

Therefore, by (4.21)(ii), we have $|L_i \cap X_j| = 1$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. And by (4.11) (b), we have $W = M$ and $W \cup X_1 \cup X_2 = L_1 \cup L_2 \cup L_3$. Hence, $|W \cap L_i| = 2$ for $i \in \{1, 2, 3\}$.

Next, we will apply Lemma 3.5 to find a 6-cluster in G traversing $L_1 \cup L_2 \cup L_3$.

Let $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{y_1, y_2, y_3\}$, $W = \{z_1, z_2, z_3\}$, let $\{x_i, y_i\} \subseteq L_i$ for $i \in \{1, 2, 3\}$, and $z_1 \in L_2 \cap L_3$, $z_2 \in L_1 \cap L_3$, $z_3 \in L_1 \cap L_2$ (by (4.2)).

Hence, we get the description of graph G as in Lemma 3.5: x_i, y_i, z_i ($1 \leq i \leq 3$) are distinct vertices of 6-connected graph G, and $L_1 = \{x_1, y_1, z_2, z_3\}$, $L_2 = \{x_2, y_2, z_3, z_1\}$, $L_3 = \{x_3, y_3, z_1, z_2\}$ are 4-cliques, and there is a partition Y_1, Y_2 of $V(G) - \{z_1, z_2, z_3\}$ with $X_1 \subseteq Y_1, X_2 \subseteq Y_2$, and x_iy_i ($1 \leq i \leq 3$) are the only edges of G with one end in Y_1 and the other in Y_2.

Therefore by Lemma 3.5, G has a 6-*cluster* traversing $L_1 \cup L_2 \cup L_3$. This is a contradiction.

4.23. We claim that, for $1 \leq j \leq n$, if $|X_j| = 3$ then $X_j = Y_j$

By (4.14), $|W| \leq 3$. By (4.15), it is obvious that $X_j = Y_j$ if $k \geq 5$ or $|W| < 3$. Hence, the only remaining case is $k = 4$ and $|W| = 3$, which was eliminated in (4.22).

4.24

(i) We claim that if $v \in A_i \cap X_j$ for some $i \in \{1, 2, 3\}$ and some $j \in \{1, 2, \ldots, n\}$, then $d_{Y_j - A_i}(v) \geq 2$, and the equality holds if and only if $d_G(v) = k + 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

By the definition of A_i (4.12), we have

$$N_G(v) - (Y_j - A_i) \subseteq A_i \cup W - \{v\}.$$

Since G is $(k + 2)$-connected and $|A_i| \leq k + 1 - |W|$ (by (4.13)(e)), we have:

$$|N_G(v) \cap (Y_j - A_i)| \geq (k + 2) - |A_i \cup W - \{v\}| \geq (k + 2) - (k + 1 - |W| + |W| - 1) = 2$$

and the equality holds if and only if $d(v) = k + 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

(ii) We claim that if $v \in A_i \cap X_j$ and $|X_j| = 3$ for some $i \in \{1, 2, 3\}$ and some $j \in \{1, 2, \ldots, n\}$, then $d_{X_j}(v) = 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

Note that $|X_j| = 3$. By (4.23), we have $Y_j = X_j$, and therefore,

$$d_{Y_j - A_i}(v) = d_{X_j - A_i}(v) \leq 2.$$

On the other hand, by (4.24)(i), we have $d_{Y_j - A_i}(v) \geq 2$. Hence, $d_{Y_j - A_i}(v) = 2$. By (4.24)(i) again, we are done.

4.25. We claim that if $|X_j| = 3$ for some j then

(i) $|X_j \cap A_i| = 1$ for each $i \in \{1, 2, 3\}$.
(ii) X_j induces a clique of G.
(iii) $W \subseteq N_G(v)$ for any $v \in X_j$.

Proof of (i): For otherwise, we may assume that $|X_j \cap A_1| \neq 1$. By (4.21)(i), $|X_j \cap A_1| = 0$, $|A_2 \cap X_j| \leq 1$ and $|A_3 \cap X_j| \leq 1$. This implies that there exists $x \in X_j$, and $x \notin A_1 \cup A_2 \cup A_3$. Since $|X_j| = 3$, we have $X_j = Y_j$ by (4.23), and by the definition of A_i (4.12), we have $N_G(x) \subseteq W \cup (Z - \{x\}) \cup (X_j - \{x\})$. Note that, by (4.20)(ii), (4.14), we have

$$|W| + |Z - \{x\}| + |X_j - \{x\}| \leq |W| + (3 + |M| - 2|W| - 1) + 2 = 4 + |M| - |W|.$$

Note that $|M| \leq |W|$ by (4.8)(a). Hence, we have $|N_G(x)| \leq 4$. This contradicts that G is $(k + 2)$-connected where $k \geq 4$ (by (4.3)).

Proof of (ii) and (iii). (ii) and (iii) are immediate corollaries of (4.25)(i) and (4.24)(ii).

4.26. We claim that there exists some $j \in \{1, 2, \ldots, n\}$ such that $|X_j| \geq 5$

By (4.19), we may assume $|X_j| = 3$ for all $j \in \{1, 2, \ldots, n\}$.

By (4.25)(i)–(ii), every vertex of $(L_1 - W) \cap X_j$ (for all $j \in \{1, 2, \ldots, n\}$) is adjacent to a vertex of $A_2 \cap X_j$ and a vertex of $A_3 \cap X_j$. By (4.25)(i) and (4.25)(iii), every vertex of $L_1 \cap W$ is adjacent to some vertex of A_2 and some vertex of A_3, hence, $\wp = \{u_1, u_2, \ldots, u_k, A_2, A_3\}$ is a $(k+2)$-*cluster* that traverses $L_1 \cup L_2 \cup L_3$ where $\{u_1, u_2, \ldots, u_k\} \in L_1$. This is a contradiction.

4.27. We claim that $|X_j| \geq 5$ for every $j \in \{1, 2, \ldots, n\}$

For otherwise, by (4.19), we may assume $|X_1| = 3$. By (4.25)(i), $|A_i \cap X_1| = 1$ for each $i \in \{1, 2, 3\}$. Hence, by (4.24)(ii), $|A_i| = k + 1 - |W|$ for each $i \in \{1, 2, 3\}$.

Furthermore, by (4.13)(b), (4.13)(c), we have

$$|Z| \geq |A_1| + |A_2| + |A_3| - |L_1 \cup L_2 \cup L_3 - W| = (3k + 3 - 3|W|) - |L_1 \cup L_2 \cup L_3 - W|.$$

However, by (4.20)(i), we have

$$|Z| = 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|.$$

The equality of (4.20)(i) implies that $|X_i| = 3$ for all $i \in \{1, 2, \ldots, n\}$. This contradicts (4.26). ■

4.28. We show some inequalities for n

By (4.27) and (4.7)(a),

$$5n \leq \sum_{1 \leq j \leq n} |X_j| \leq 2 * (k + 1 - |W|) + n = 2k + 2 + n - 2|W|. \tag{4}$$

The inequality (4) can be simplified as

$$2n \leq k + 1 - |W|. \tag{5}$$

Note that the equality (4) (and (5), as well) holds if and only if $|X_i| = 5$ for every i.

4.29. We claim that $n = k - 2$

For otherwise, since $n \geq k - 2$ by (4.11)(a), we may assume that $n \geq k - 1$.

By (5), we have

$$2k - 2 \leq 2n \leq k + 1 - |W|. \tag{6}$$

That is,

$$k \leq 3 - |W|.$$

Note that $k \geq 4$ by (4.3). This is a contradiction.

4.30. The final step of the proof

By (4.29), $n = k - 2$. By (4.11)(b), we have

$$W = M \quad \text{and} \quad L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n.$$

Hence,

$$Z = \emptyset. \tag{7}$$

By (5) of (4.28), we have

$$2k - 4 = 2n \leq k + 1 - |W|.$$

That is,

$$k \leq 5 - |W|. \tag{8}$$

Note that $k \geq 4$ by (4.3). Therefore, there are only two cases: $k = 5$ and $k = 4$ (by (7) and (8)).

Case 1: $k = 5$. In this case, $|W| = 0$. By (4.29), $n = k - 2 = 3$. The equality of (5) of (4.28) implies that

$$|X_1| = |X_2| = |X_3| = 5.$$

By (4.21)(ii), without loss of generality, we assume $|L_1 \cap X_1| = 2$. By (4.18), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 6 since $Z = \emptyset$ and $W = \emptyset$. This contradicts that G is $(k+2)$-connected where $k = 5$.

Case 2: $k = 4$. In this case, $n = k - 2 = 2$ (by (4.29)). There are two subcases: $|W| = 1$ and $|W| = 0$ (by (8)).

Subcase 1: $|W| = 1$. The equality (5) of (4.28) implies that

$$|X_1| = |X_2| = 5.$$

Without loss of generality, we assume $W \subseteq L_1$ and $|L_1 \cap X_1| = 2$. By (4.18), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 5 since $Z = \emptyset$ and $W \subseteq L_1$. This contradicts that G is $(k+2)$-connected where $k = 4$.

Subcase 2: $|W| = 0$.

Since $Z = \emptyset$ by (7), we have

$$\sum_{j=1}^{n} |X_j| = |L_1 \cup L_2 \cup L_3| = 3k = 12. \tag{9}$$

Therefore, the only possibility in this subcase is that $|X_1| = 5$ and $|X_2| = 7$ (by (4.10) and (4.27)).

Without loss of generality, we assume $|L_1 \cap X_1| = 2$. By (4.18), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 5 since $Z = \emptyset$ and $W = \emptyset$. This contradicts that G is $(k+2)$-connected where $k = 4$.

This completes the proof of main theorem.

5. Proof of Lemma 3.5

The following two lemmas will be useful in the proof of Lemma 3.5.

Lemma 5.1 (*Whitney [4]*). *Any two planar embeddings of a 3-connected graph are equivalent.*

Let $G = (V, E)$ is a graph and $A \subseteq V(G)$, we denote the set of vertices on $V(G) - A$ which are adjacent to some vertex in A by $\Upsilon(A)$. That is $\Upsilon(A) = N(A) - A$.

Lemma 5.2 (*Seymour [16], Thomassen [18]*). *Let s_1, t_1, s_2, t_2 be distinct vertices of a graph $G = (V, E)$. Then just one of the following is true:*

(i) *there are paths joining s_1 to t_1 and s_2 to t_2 respectively, vertex-disjoint.*
(ii) *for some $k \geq 0$ there are pairwise disjoint sets $A_1, \ldots, A_k \subseteq V(G) - \{s_1, s_2, t_1, t_2\}$ such that*
  (a) *for $i \neq j$, $\Upsilon(A_i) \cap A_j = \emptyset$,*
  (b) *for $1 \leq i \leq k$, $|\Upsilon(A_i)| \leq 3$,*
  (c) *if $\widetilde{G}$ is the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $\Upsilon(A_i)$, and also for $j = 1, 2$ adding an edge e_j joining s_j to t_j, then $\widetilde{G}$ may be drawn in the plane with no pairs of edges crossing except e_1, e_2 which cross once.*

Proof of Lemma 3.5. Let G be a counterexample to the lemma with least number of vertices. Let $W = \{z_1, z_2, z_3\}$, $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{y_1, y_2, y_3\}$. Let $\Im_{ij} = \{z_i\} \cup X_j$ and G_{ij} be the subgraph induced by $\{z_i\} \cup Y_j$ where $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Let f_1, f_2, f_3 be the edges with ends z_2z_3, z_3z_1, and z_1z_2 respectively.

5.1. *We claim that $G - \{f_1, f_2, f_3\}$ is not planar*

Since G is 6-connected, $|E(G)| \geq 3|V(G)|$. Therefore $|E(G - \{f_1, f_2, f_3\})| \geq 3|V(G)| - 3$. Note that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Hence $G - \{f_1, f_2, f_3\}$ is not planar.

5.2

The strategy of the proof is to prove that $G' = G - \{f_1, f_2, f_3\}$ is planar (hence contradicts 5.1). The planarity of G' is yielded by showing that each $G_i = G'[W \cup Y_i](1 \leq i \leq 2)$ has a planar embedding with vertices $z_1, x_2, z_3, x_1, z_2, x_3$ for $i = 1$ (or $z_1, y_2, z_3, y_1, z_2, y_3$ for $i = 2$, respectively) around its exterior face (Lemma 5.2 is applied here). And the planar embedding of G_i is constructed from the unique embedding of the 3-connected graph $H_i = Y_i \cup E_i$ $(1 \leq i \leq 2)$ where $E_1 = \{x_1x_2, x_1x_3, x_2x_3\}$, and $E_2 = \{y_1y_2, y_1y_3, y_2y_3\}$, and planar embeddings of other subgraphs of $G_i \cup E_i$.

5.3. *We claim that $|Y_1|, |Y_2| \geq 4$*

For otherwise, we may assume $|Y_1| = 3$. That is, $Y_1 = X_1$. Since G is 6-connected, $d_G(x_1) \geq 6$. Hence $W \cup \{y_1, x_2, x_3\} \subseteq N_G(x_1)$. With the similar argument, we have $W \cup \{y_3, x_1, x_2\} \subseteq N_G(x_3)$ and $W \cup \{y_2, x_1, x_3\} \subseteq N_G(x_2)$. Therefore, $\wp = \{\{x_1\}, \{x_2\}, \{x_3\}, \{z_1\}, \{z_2\}, \{z_3\}\}$ is a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

5.4. We claim that the subgraph induced by Y_i is connected for $i = 1, 2$

For otherwise, we assume that the subgraph induced by Y_1 is not connected. Let D be a component not containing x_1, then $\{x_2, x_3, z_1, z_2, z_3\}$ is a 5-cutset separating D and Y_2, it contradicts that G is 6-connected.

5.5. We claim that there is no 4-cluster in G_{ij} traversing $\Im_{ij}$ where $1 \leq i \leq 3$, $1 \leq j \leq 2$

For otherwise, we may assume that there is a 4-cluster $\wp_1$ in G_{11} traversing $\Im_{11}$. Let $\wp = \wp_1 \cup \{\{z_2, z_3\}, \{Y_2\}\}$, then $\wp$ is a 6-cluster in G traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

5.6. We claim that both X_1 and X_2 are independent sets of G

For otherwise, without loss of generality, we assume that $x_1x_2 \in E(G)$. There is a vertex $x \in Y_1 - X_1$ by (5.3) and there are 4 internally disjoint paths $P_{x_1}, P_{x_2}, P_{x_3}, P_{z_3}$ of $G - \{z_1, z_2\}$ since $G - \{z_1, z_2\}$ is 4-connected where P_u is the path joining x and $u \in \{x_1, x_2, x_3, z_3\}$.

Now since x_iy_i $(1 \leq i \leq 3)$ are the only edges of G with one end in Y_1 and the other in Y_2, we have $P_u \cap Y_2 = \emptyset$ where $u \in \{x_1, x_2, x_3, z_3\}$. Hence $\wp = \{P_{x_1} - x, P_{x_2} - x, P_{z_3} - x, P_{x_3}\}$ is a 4-cluster in G_{31} traversing $\Im_{31}$. This contradicts (5.5).

5.7. We claim that there are no two vertex-disjoint paths in G_{11} joining z_1 to x_1 and x_2 to x_3 respectively

For otherwise, let P_1 and P_2 be the paths joining z_1 to x_1 and x_2 to x_3 respectively. Let A_1 and A_2 be disjoint fragments of the subgraph induced by Y_1 with $P_1 - \{z_1\} \subseteq A_1$, $P_2 \subseteq A_2$, and $A_1 \cup A_2$ maximal. Since the subgraph induced by Y_1 is connected by (5.4), A_1 and A_2 are adjacent. Therefore $\wp = \{\{z_1\}, \{z_2\}, \{z_3\}, A_1, A_2, Y_2\}$ is a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

5.8

(i) We claim that G_{11} can be drawn in a plane so that every vertex in $\Im_{11}$ is incident with the infinite region and z_1, x_2, x_1, x_3 are around its exterior face in this order, and this embedding is denoted by π_{11}.

By Lemma 5.2 and (5.7), there exists some $k \geq 0$ and there are pairwise disjoint sets $A_1, \ldots, A_k \subseteq V(G) - \{z_1, x_2, x_1, x_3\}$ such that

(a) for $i \neq j$, $\Upsilon(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|\Upsilon(A_i)| \leq 3$,

(c) if $\widetilde{G_{11}}$ is the graph obtained from G_{11} by deleting A_i and adding new edges joining every pair of distinct vertices in $\Upsilon(A_i)$, and adding an edge e_1 joining z_1 to x_1 and an edge e_2 joining x_2 to x_3, then $\widetilde{G_{11}}$ may be drawn in the plane with no pairs of edges crossing except e_1, e_2 which cross once.

We claim that $k = 0$. For otherwise, $\Upsilon(A_k) \cup \{z_2, z_3\}$ is a cut set of order at most 5 separating A_k and Y_2, it contradicts that G is 6-connected.

Hence we have $G_{11} = \widetilde{G_{11}} - \{e_1, e_2\}$, and result follows.

With the similar argument, we have

(ii) G_{21} can be drawn in a plane so that z_2, x_1, x_2, x_3 are around its exterior face in this order, and this embedding is denoted by π_{21}.

(iii) G_{31} can be drawn in a plane so that z_3, x_1, x_3, x_2 are around its exterior face in this order, and this embedding is denoted by π_{31}.

5.9. Let G_i be the graph obtained from the subgraph induced by $W \cup Y_i$ and deleting edges f_1, f_2, f_3 for $1 \leq i \leq 2$, and let G_1^+ be the graph obtained from G_1 by adding edges x_1x_2, x_1x_3, x_2x_3, let G_2^+ be the graph obtained from G_2 by adding edges y_1y_2, y_1y_3, y_2y_3

In next few subsections, we are to show that G_1^+ has a planar embedding such that the triangle $x_1x_2x_3x_1$ is a facial circuit.

5.10

(i) Let H_{h1} be the graph obtained from G_1^+ by deleting z_i and z_j where $\{h, i, j\} = \{1, 2, 3\}$. Obviously H_{h1} is also the graph obtained from G_{h1} by adding edges x_1x_2, x_1x_3, x_2x_3. By (5.8), G_{h1} has an embedding π_{h1} with $\{x_i, x_h, x_j, z_h\}$ in its exterior face. Hence π_{h1} can also be considered as an embedding of H_{h1} with the triangle $x_1x_2x_3$ as the exterior face. We denote this embedded graph by $\pi_{h1}(H_{h1})$.

(ii) Let H_1 be the graphs obtained from the subgraph induced by Y_1 by adding edges x_1x_2, x_1x_3, x_2x_3. Obviously, H_1 can also be obtained from H_{h1} by deleting z_h for $h \in \{1, 2, 3\}$. Hence π_{h1} can also be considered as an embedding of H_1, and we denote this embedded graph by $\pi_{h1}(H_1)$.

5.11. We claim that H_1 is a 3-connected planar graph

Planarity is an immediate conclusion from (5.10)(ii). Next we will show that H_1 is 3-connected.

For otherwise, we assume that H_1 is not 3-connected. Let A be a cut set of order less than 3. Since $\{x_1, x_2, x_3\}$ induces a clique in the graph H_1, let D be a component of $H_1 - A$ with $x_i \notin D$ for every $i \in \{1, 2, 3\}$. Then $A \cup \{z_1, z_2, z_3\}$ is a cutset of G of order at most 5 separating D and Y_2, it is a contradiction.

5.12. By Lemma 5.1, H_1 has only one embedding π_0. That is, $\pi_0(H_1) = \pi_{h1}(H_1)$ for each $h \in \{1, 2, 3\}$.

5.13. (i) We claim that G_1^+ has a planar embedding such that the triangle $x_1x_2x_3x_1$ is a facial circuit

Let C_{ij} be the facial circuit of the embedded graph $\pi_0(H_1)$ containing the edge x_ix_j other than the triangle $x_1x_2x_3x_1$. Let F_{ij} be the face of $\pi_0(H_1)$ bounded by C_{ij} where $1 \le i < j \le 3$.

By (5.12), $\pi_0(H_1) = \pi_{11}(H_1)$, and by (5.10) (ii), $\pi_{11}(H_1)$ is obtained from $\pi_{11}(H_{11})$ by deleting the vertex z_1, hence z_1 must be inside the face F_{23} of $\pi_0(H_1)$, and all neighborhoods of z_1 must be in the facial circuit C_{23}. Similarly, for each $\{h, i, j\} = \{1, 2, 3\}$, all neighborhoods of z_h must be in the facial circuit C_{ij}.

Note that F_{ij}'s are distinct for $1 \le i < j \le 3$ since H_1 is 3-connected by (5.11). Now we can get a planar embedding of G_1^+ by adding the vertex z_h and edges z_hu into the face F_{ij} where $u \in N_G(z_h) \cap Y_1$.

With the similar argument, we have

(ii) G_2^+ has a planar embedding such that the triangle $y_1y_2y_3y_1$ is a facial circuit.

5.14. The final step of the proof

By (5.13) and (5.9), G_i is planar with triangle $z_1z_2z_3$ as the exterior face.

Now we identify z_i of G_1 to z_i of G_2 where $1 \le i \le 3$, and join x_i of G_1 to y_i of G_2, we get planar graph $G - \{f_1, f_2, f_3\}$, it contradicts (5.1).

This completes the proof of this lemma. ∎

Acknowledgement

The second author's research was partially supported by the National Security Agency under Grant MDA 904-01-1-0022.

References

[1] K. Appel, W. Haken, Every planar map is four colorable, Part I. Discharging, Illinois J. Math. 21 (1977) 429–490.
[2] K. Appel, W. Haken, J. Koch, Every planar map is four colorable, Part II. Reducibility, Illinois J. Math. 21 (1977) 491–567.
[3] G. Chartrand, D.P. Geller, T. Hedetniemi, Graphs with forbidden subgraphs, J. Combin. Theory 10 (1971) 12–41.
[4] Reinhard Diestel, Graph Theory, 2nd ed., Springer-Verlag, New York, 1997, pp. 79–80.
[5] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[6] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljahrsschr. Naturforsch. Ges. Zürich 88 (1943) 133–142.
[7] L.K. Jørgensen, Contractions to K_8, J. Graph Theory 18 (1994) 431–448.
[8] K. Kawarabayashi, Minors in 7-chromatic graphs, preprint.
[9] K. Kawarabayashi, R. Luo, J. Niu, C.-Q. Zhang, On structure of k-connected graphs without K_k-minor, Europ. J. Combin. 26 (2005) 293–308.
[10] K. Kawarabayashi, B. Toft, Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor, Combinatorica 25 (2005) 327–353.
[11] W. Mader, Homomorphiesätze für Graphen, Math. Ann. 178 (1968) 154–168.
[12] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math Ann. 174 (1967) 265–268.
[13] W. Mader, Über die Maximalzahl kreuzungsfreier H-Wege, Arch. Math 31 (1978) 387–402.
[14] N. Robertson, D.P. Sanders, P.D. Seymour, R. Thomas, The four-color theorem, J. Combin. Theory Ser. B. 70 (1997) 2–44.
[15] N. Robertson, P.D. Seymour, R. Thomas, Hadwiger conjecture for K_6-free graphs, Combinatorica 13 (1993) 279–361.
[16] P.D. Seymour, Disjoint paths in graphs, Combinatorica 29 (1980) 293–309.
[17] Z. Song, R. Thomas, The extremal function for K_9-minor, J. Combin. Theory Ser. B. 96 (2006) 240–252.
[18] C. Thomassen, 2-linked graphs, Europ. J. Combin. 1 (1980) 371–378.
[19] B. Toft, On separating sets of edges in contraction-critical graphs, Math Ann. 196 (1972) 129–147.
[20] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.
[21] D.R. Woodall, Improper colourings of graphs, in: Graph Colourings, in: Pitman Research Notes, vol. 218, Longman, 1990, pp. 45–63.