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Discrete Mathematics 301 (2005) 232–238
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A note about shortest cycle covers

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Received 5 August 2003; received in revised form 2 May 2005; accepted 29 June 2005
Available online 19 August 2005

Abstract

Let G be a graph with odd edge-connectivity r. It is proved in this paper that if $r>3$, then G has a 3-cycle $(1,2)$-cover of total length at most $((r+1)|E(G)|)/r$.

Keywords: Cycle cover; Shortest cycle cover; Odd-edge connectivity; r-graph

1. Introduction

Let $G=(V,E)$ be a bridgeless graph and $\mathscr{C}$ be a cycle cover of G. Denote the total length of the cycle cover $\mathscr{C}$ by $\ell(\mathscr{C})$. A cycle cover of G with the minimum total length is called a *shortest cycle cover* of G, and its total length is denoted by $\mathrm{SCC}(G)$.

Extensive studies have been done to estimate the upper bound of $\mathrm{SCC}(G)$. There were two major conjectures in this area, one estimates that $\mathrm{SCC}(G)\leqslant \frac{7}{5}|E|$ (Conjecture 1.1) and another one estimates that $\mathrm{SCC}(G)\leqslant |E|+|V|-1$ (originally conjectured by Itai and Rodeh in [17], and recently proved by Fan in [12]).

Conjecture 1.1 (*Alon and Tarsi [1]*). Let $G=(V,E)$ be a bridgeless graph. Then the total length of a shortest cycle cover of G is at most $7|E(G)|/5$.

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[1] Partially supported by National Natural Science Foundation of China under Grants no. 19671029 and 10271048 and Shanghai Priority Academic Discipline.

[2] Partially supported by the National Security Agency under Grants MDA904-01-1-0022 and MSPR-03G-023.

doi:10.1016/j.disc.2005.06.013

An *odd-edge-cut* of a graph G is an edge-cut of odd size and the *odd edge-connectivity* of G is the size of a smallest odd-edge-cut.

The following is our main theorem in this paper.

Theorem 1.2. *Let G be a graph with odd edge-connectivity r. If $r > 3$, then G has a 3-cycle $(1, 2)$-cover of total length at most $(r + 1)|E(G)|/r$.*

Theorem 1.2 is an approach to Conjecture 1.1 which strengthens an early theorem by Jackson [18] (see Theorem 2.1-3).

Remark. How about graphs with odd edge-connectivity $r = 3$? Note that the Petersen graph does not have a 3-cycle $(1, 2)$-cover (see [32]). However, for any bridgeless graph G without Petersen minor, it can be shown (by following the proof of Theorem 1.2 and applying Theorem 4.6) that G has a cycle $(1, 2)$-cover of total length at most $4|E(G)|/3$. This corollary improves the $\mathrm{SCC}(G)$ bound found by Bermond et al. [4] and Alon and Tarsi [1] for a certain family of graphs (see Theorem 2.1-4).

2. Early results and open problems

The following is a partial list of some early results.

Theorem 2.1. *Let $G = (V, E)$ be a bridgeless graph. Then G has a cycle cover $\mathscr{C}$ such that*

1. (Raspaud [22]) $\ell(\mathscr{C}) \leqslant |V| + |E| - 3$ *if G is simple and $G \neq K_4$ and admits a nowhere-zero 4-flow (several related results can be seen in* [31,34,30]).
2. (Fan [12]) $\ell(\mathscr{C}) \leqslant |E| + |V| - 1$ (*this was originally a conjecture by Itai and Rodeh* [17]. *Several related results can be seen in* [15,8], etc.).
3. (Jackson [18]) $\ell(\mathscr{C}) \leqslant (2m + 2)/(2m + 1)|E|$ *if $m \geqslant 2$ and G is $2m$-edge-connected*.
4. (Bermond et al. [4,1]) $\ell(\mathscr{C}) \leqslant \frac{5}{3}|E|$.
5. (Jamshy et al. [20]) $\ell(\mathscr{C}) \leqslant \frac{8}{5}|E|$ *if G admits a nowhere-zero 5-flow*.
6. (Fan [9]) $\ell(\mathscr{C}) \leqslant \frac{7}{5}|E|$ *if Tutte's 3-flow conjecture is true*.
7. (Fan and Raspaud[13]) $\ell(\mathscr{C}) \leqslant \frac{22}{15}|E|$ *if Fulkerson's conjecture is true*.

Furthermore, it was proved by Thomassen [26] that finding a shortest cycle cover of a bridgeless graph is an NP-complete problem and it was proved by Jamshy and Tarsi [21] that Conjecture 1.1 implies the well-known *cycle double cover conjecture* [25,23].

3. Notation and terminology

Commonly used graph theory notation and terminology are not to be defined in this paper. Readers are referred to [5,29,6].

A *circuit* is a connected 2-regular subgraph and a *cycle* is the union of edge-disjoint circuits.

A subgraph P of a graph G is called a *parity subgraph* of G if $d_P(v) \equiv d_G(v) \pmod 2$ for every vertex v of G.

Let G be a graph. The *underlying graph* of G, denoted by $\overline{G}$, is the graph obtained from G by replacing every maximal induced path with a single edge.

An r-regular graph is called an *r-graph* if, for every $X \subseteq V(G)$ of odd order, the number of edges of G with one end in X is at least r.

Let D be an orientation of $E(G)$ and $f : E(G) \mapsto Z$. The ordered pair (D, f) is an *integer flow* of G if $\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$ for every vertex v of G. An integer flow (D, f) is called a *k-flow* if $|f(e)| < k$ for every edge e of G. An integer flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every edge e of G. (The concept of integer flow was introduced by Tutte in [27,28] as a generalization of map coloring problem.)

Let $w : E(G) \mapsto Z^+$. The weight w is *eulerian* of G if the total weight of every edge-cut is even. The total weight of a subgraph H of G is denoted by $w(H)$.

Let w be an eulerian weight of a graph G. A cycle cover $\mathscr{C}$ of G is *faithful* with respect to w if every edge e of G is contained in precisely $w(e)$ members of the family $\mathscr{C}$. If w is a constant 2 weight, a faithful cycle cover with respect to w is a *cycle double cover* of G. A family $\mathscr{C}$ of cycles is a $(1, 2)$-*cover of G* if every edge of G is contained in precisely one or two members of $\mathscr{C}$.

A cycle cover $\mathscr{C}$ of G is a *k-cycle cover* of G if $|\mathscr{C}| \leqslant k$.

4. Major theorems and lemmas applied in the proofs

Lemma 4.1 (*Zhang* [33]). *Let $G=(V, E)$ be a graph with odd edge connectivity r. Assume that there is a vertex $v \in V(G)$ such that $d(v) \neq r$ and $\neq 2$. Let $E(v) = \{e_1, \ldots, e_d\}$ with x_i be the end vertex of e_i other than v. Then there is a pair of edges e_i and $e_j \in E(v)$ such that the graph obtained from G by deleting e_i and e_j and adding a new edge joining x_i and x_j remains of odd edge connectivity r.*

Theorem 4.2 (*Edmonds* [7], *or see Seymour* [24]). *Let G be an r-graph. Then G has a family $\mathscr{M}$ of perfect matchings such that there is an integer p, each edge of G is contained in precisely p members of $\mathscr{M}$.*

Theorem 4.3 (*Jaeger* [19]). *Every* 4-*edge-connected graph admits a nowhere-zero* 4-*flow*.

Assume that $\{e, e'\}$ is the 2-edge-cut of a graph G. It is (well-known and) easy to see that G/e admits a nowhere-zero k-flow if and only if G admits a nowhere-zero k-flow. Hence, by ignoring 2-edge-cuts of graphs, we can restate Theorem 4.3 as follows.

Corollary 4.4. *Every graph with odd edge-connectivity at least five admits a nowhere-zero* 4-*flow*.

Theorem 4.5 (*Goddyn* [16], *Zhang* [31]). *If a graph G admits a nowhere-zero* 4-*flow, then G has a faithful* 3-*cycle cover with respect to every* $(1, 2)$-*eulerian weight*.

Theorem 4.6 (*Alspach et al.* [2]). *Let G be a bridgeless graph containing no Petersen minor. Then G has a faithful circuit cover with respect to every* $(1, 2)$*-eulerian weight.*

5. Proof of the main theorem

Lemma 5.1. *Let r be an odd integer and G be an r-regular graph. Then G is an r-graph if and only if the odd edge-connectivity of G is r.*

Proof. "$\Rightarrow$": Assume that G is an r-graph. Let T be an odd edge-cut of G. Let X and Y be two components of $G - T$. That is, $T = [X, Y]$. Contracting one component Y of $G - T$, we obtain a graph G' consisting of all odd degree vertices since one of them, created by the contraction, is of degree $|T|$, while all others, vertices of X, remain of degree r. Since the number of odd degree vertices in G' must be even, we have that $|X|$ is odd. By the definition of r-graph, the edge-cut T must be of order at least r. Hence, every odd edge-cut of G must be of size at least r.

"$\Leftarrow$": Assume that the odd edge connectivity of G is r. Let $X \subseteq V(G)$ of odd order. With a similar argument as above by considering the parity of the number of odd vertices in a graph, we can prove that the edge-cut $[X, V(G) - X]$ is of odd size. Hence, by the definition of odd edge-connectivity, the edge cut $[X, V(G) - X]$ must be of size at least r. Hence, G is an r-graph. □

Lemma 5.2. *Let G be a graph with odd edge-connectivity r. Then there is a series of edge-splitting operations such that the resulting graph* G' *by applying those operations on G has the following property,*

(1) *the odd edge-connectivity of* G' *remains r;*
(2) $d_{G'}(v) = r$ *or* 2 *for every vertex of* $v \in V(G')$;
(3) *the underlying graph* $\overline{G'}$ *of* G' *is an r-graph.*

Proof. By Lemma 5.1, we need to prove only the assertion (1). Hence, assume that there is a vertex $v \in V(G)$ with $\mathrm{d}(v) \neq r$ and 2. By applying Lemma 4.1, one can split two edges away from v by maintaining the odd-edge-connectivity. Repeat this procedure until all vertices of the resulting graph is of degree either r or 2. □

Note that, by Lemma 4.1, we can obtain an r-graph from a graph with odd edge-connectivity at least r. The new graph (r-graph), furthermore, preserves the embedding property of G (see [33]. However, this embedding property is not needed in this paper.)

If the graph G that we worked in Lemma 5.2 is a weighted graph (that is, each edge has a length). One may define a one-to-one-mapping h between $E(G)$ and $E(G')$ in the natural way. That is, $h(e) = e$ since the labels of edges are not changed in the splitting operation. (However, the corresponding relation between the edges of G and $\overline{G'}$ may not be a one-to-one relation since each maximal induced path of G' becomes a single edge in $\overline{G'}$.)

It was proved in [18] that each $(2m + 1)$-edge-connected graph G has a parity subgraph of size at most $1/(2m + 1)|E(G)|$. Lemma 5.3 strengthens this result by replacing edge-connectivity with odd-edge-connectivity.

Lemma 5.3. *Let G be a graph with odd edge-connectivity r, and* $w : E(G) \mapsto Z^+$ *be a non-negative weight* (*length of edges*). *Then G contains a parity subgraph P with* $w(P) \leqslant w(G)/r$.

Proof. By Lemma 5.2, there is a series of splitting operations of G such that the underlying graph of the resulting graph G' is an r-graph.

Define $h : E(G) \mapsto E(\overline{G'})$ such that, for each $e \in E(\overline{G})$, $h^{-1}(e)$= all edges of the corresponding induced path of G' and define w' for $\overline{G'}$ as follows: $w'(e) = \sum_{e' \in h^{-1}(e)} w(e)$.

By Theorem 4.2, let $\mathscr{M}$ be the set of 1-factors covering each edge of G' precisely p times. Since $|\mathscr{M}| = rp$ and

$$\sum_{M \in \mathscr{M}} w(M) = p \times w(G'),$$

there is a 1-factor $M_0 \in \mathscr{M}$ with

$$w(M_0) \leqslant \frac{1}{r} w(G').$$

Note that the subgraph of G induced by edges of $h^{-1}(M_0)$ is a parity subgraph of G of total weight at most $(1/r)w(G)$. □

The Proof of the Main Theorem. By Lemma 5.3, let P be a parity subgraph of G with $w(P) \leqslant w(G)/r$. Define $f : E(G) \mapsto \{1, 2\}$ as follows:

$$f(e) = \begin{cases} 2 & \text{if } e \in E(P), \\ 1 & \text{if } e \notin E(P). \end{cases} \tag{1}$$

It is not hard to see that f is an eulerian weight of G: for each $v \in V(G)$,

$$\begin{aligned} \sum_{e \in E(v)} f(e) &= |E(v) - E(P)| + 2|E(v) \cap E(P)| \\ &= |E(v)| + |E(v) \cap E(P)| \\ &\equiv \mathrm{d}(v) + \mathrm{d}(v) \equiv 0 \bmod (2). \end{aligned}$$

If one can find a faithful cycle cover $\mathscr{F}$ of G with respect to f, then, obviously,

$$\sum_{C \in \mathscr{F}} w(C) = w(G) + w(P) \leqslant \frac{(r+1) \times w(G)}{r}.$$

So, we are to find the faithful cycle cover $\mathscr{F}$ in two cases.

If $r \geqslant 5$, then, by Corollary 4.4, G admits a nowhere-zero 4-flow. By Theorem 4.5, the graph G has a faithful cycle cover $\mathscr{F}$ with respect to the $(1, 2)$-eulerian weight f defined in (1), and furthermore, $|\mathscr{F}| \leqslant 3$. □

6. Remarks

Conjecture 6.1. Let G be a graph with odd edge-connectivity r. If $r > 3$, then G has a 2-cycle $(1, 2)$-cover of total length at most $((r+1)|E(G)|)/r$.

Conjecture 6.1 is a generalization of an open problem proposed by Fan.

Conjecture 6.2 (*Fan [8]*). Every 4-edge-connected graph G has a 2-cycle cover $\mathscr{C}$ with $\ell(\mathscr{C}) \leqslant \frac{6}{5}|E|$.

It is not hard to see that Conjecture 6.1 is an corollary of the following conjecture by Seymour.

Conjecture 6.3 (*Seymour [24]*). For each $r \geqslant 4$, every r-graph G has a perfect matching M such that $G - E(M)$ is an $(r-1)$-graph.

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