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On flows in bidirected graphs

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Abstract

Bouchet conjectured that every bidirected graph which admits a nowhere-zero bidirected flow will admit a nowhere-zero bidirected 6-flow [A. Bouchet, Nowhere-zero integer flows on a bidirected graph, J. Combin. Theory Ser. B 34 (1983) 279–292]. He proved that this conjecture is true with 6 replaced by 216. Zyka proved in his Ph.D dissertation that it is true with 6 replaced by 30. Khelladi proved it is true with 6 replaced with 18 for 4-connected graphs [A. Khelladi, Nowhere-zero integer chains and flows in bidirected graphs, J. Combin. Theory Ser. B 43 (1987) 95–115]. In this paper, we prove that Bouchet's conjecture is true for 6-edge connected bidirected graphs.

Keywords: Bidirected graphs; Integer flows; Bidirected *k*-flows

1. Introduction

A bidirected graph G is a directed graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge is oriented as one of the four possibilities: $\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\blacktriangleright\!\!-\!\!\bullet$, $\bullet\!\!-\!\!\blacktriangleleft\!\!-\!\!\blacktriangleleft\!\!-\!\!\bullet$, $\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\blacktriangleleft\!\!-\!\!\bullet$, $\bullet\!\!-\!\!\blacktriangleleft\!\!-\!\!\blacktriangleright\!\!-\!\!\bullet$. An edge with orientation $\bullet\!\!-\!\!\blacktriangleright\!\!-\!\!\blacktriangleleft\!\!-\!\!\bullet$ (resp., $\bullet\!\!-\!\!\blacktriangleleft\!\!-\!\!\blacktriangleright\!\!-\!\!\bullet$) is called an *in-edge* (resp., *out-edge*). Edges with other orientations are called *ordinary edges*. The set of in(out)-edges is denoted by $E_{\mathrm{i}}(G)$ ($E_{\mathrm{o}}(G)$, respectively). We define $\mathrm{O}(G) = |E_{\mathrm{i}}(G)| + |E_{\mathrm{o}}(G)|$.

For an ordinary edge e, reversing the orientation of e means the natural way to change its orientation. For any non-ordinary edge $e \in E(G)$, reversing the orientation of e means

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changing *e* from an in(out)-edge to an out(in)-edge. Note that after reversing the orientation of an edge of G, $\mathrm{O}(G)$ remains the same.

Let G be a bidirected graph. For any $v \in V(G)$, the set of all edges with tails (or heads) at v is denoted by $E^+(v)$ (or $E^-(v)$) and we define $E(v) = E^+(v) \cup E^-(v)$. (For a general graph G, we use $E(v)$ to denote the set of edges which are incident with v in G.) A bidirected graph is eulerian if $|E(v)|$ is even for each $v \in V(G)$. Readers are referred to [1] for terminology not defined in this paper.

Definition 1.1. Let G be a bidirected graph and f be a function: $E(G) \mapsto Z$. Then:

(1) f is called a *bidirected k-flow* of G if $-k+1 \leqslant f(e) \leqslant k-1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$;
(2) f is called a *bidirected modular k-flow* of G if $0 \leqslant f(e) \leqslant k-1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod k$ for every $v \in V(G)$;
(3) The *support* of a bidirected k-flow (resp., modular k-flow) f of G is the set of edges of G with $f(e) \neq 0$ (resp., $f(e) \not\equiv 0 \pmod k$), and is denoted by $\mathrm{supp}(f)$. A bidirected k-flow (modular k-flow) of G is *nowhere-zero* if $\mathrm{supp}(f) = E(G)$.

For nowhere-zero bidirected integer flows, Bouchet [2] proposed the following conjecture (see also Toft and Jensen's book [11]).

Conjecture 1.2. *Every bidirected graph which admits a nowhere-zero bidirected flow will admit a nowhere-zero bidirected* 6-*flow*.

Note that the value 6 in Conjecture 1.2 is best possible, see [2] for details. The following is a list of the partial results to Conjecture 1.2.

Theorem 1.3. *Let G be a bidirected graph admitting a nowhere-zero bidirected flow. Then*:

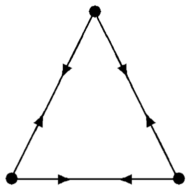
(1) (*Bouchet* [2]) *G admits a nowhere-zero bidirected* 216-*flow*;
(2) (*Zyka* [16], *or see* [6]) *G admits a nowhere-zero bidirected* 30-*flow*;
(3) (*Khelladi* [6]) *G admits a nowhere-zero bidirected* 18-*flow if G is* 4-*connected*.

Bidirected flow is a generalization of the concept of integer flow (introduced by Tutte [12,15] as a dual version for the vertex coloring problem). This is because a directed graph G can be considered as a bidirected graph G^* with $\mathrm{O}(G^*) = 0$. However, bidirected flows and integer flows can be quite different due to the existence of the in-edges and out-edges. Some results for integer flows can be generalized to bidirected flows while some other results cannot. The following observation for bidirected 2-flows is a generalization of Tutte's 2-flow characterization. Though the proof for integer 2-flows is straightforward, the corresponding result for bidirected 2-flow does need a few more steps in the proof.

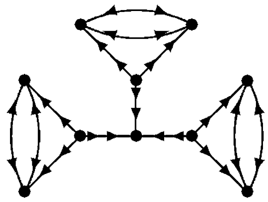
Proposition 1.4. *Let G be a connected bidirected graph. Then, G admits a nowhere-zero bidirected* 2-*flow if and only if G is a connected eulerian graph and* $\mathrm{O}(G)$ *is even*.

We will provide a proof of this proposition in Section 2.

Integer flows and modular flows are proved to be equivalent for general graphs (see [12,13] or [15]). However, for bidirected graphs, they are not equivalent to each other in many cases. The following are some examples that a nowhere-zero modular 2-flow ($f(e) = 1$ for each $e \in E(G)$) in Example 1 and a nowhere-zero modular 3-flow in Example 2 ($f(e) = 1$ for each $e \in E(G)$) cannot be converted to a nowhere-zero bidirected 2-flow (or 3-flow). The fact that the graph in Example 2 has no bidirected 3-flow will follow from Lemma 3.2. For nowhere-zero bidirected modular 3-flows, we are able to establish their equivalent relation for 2-edge connected bidirected graphs.



Example 1



Example 2

The following is one of our main results in this paper which provides a major step in the proof of Theorem 1.6.

Theorem 1.5. *Let G be a* 2*-edge connected bidirected graph. Then G admits a nowhere-zero bidirected* 3*-flow if and only if G admits a nowhere-zero modular bidirected* 3*-flow.*

By applying Theorem 1.5, we are able to verify Conjecture 1.2 for 6-edge connected bidirected graphs.

Theorem 1.6. *Let G be a* 6*-edge connected bidirected graph. Then G admits a nowhere-zero bidirected flow if and only if it admits a nowhere-zero bidirected* 6*-flow.*

2. Proof of Proposition 1.4 and its applications

A *circuit* is a connected 2-regular graph and a *cycle* is a graph such that every vertex is of even degree. Let G be a bidirected graph. A circuit C of G is said to be *balanced* provided that $\mathrm{O}(C)$ is even and *unbalanced* otherwise. A cycle P of G is said to be *balanced* provided that for each of its components P', $\mathrm{O}(P')$ is even. A collection of cycles $F = \{P_1, P_2, \ldots, P_r\}$ of G is called a *proper r-cycle cover* of G if every P_i is balanced and $\bigcup_{i=1}^{r} E(P_i) = E(G)$.

Proof of Proposition 1.4. "$\Longrightarrow$". Note that if f is a nowhere-zero bidirected 2-flow of G and we reverse the orientation of some edge e_1, then the resulting graph G_1 admits a nowhere-zero bidirected 2-flow f_1, such that, $f_1(e) = f(e)$ for any $e \neq e_1$ and $f_1(e_1) = -f(e_1)$. Also, $\mathrm{O}(G) = \mathrm{O}(G_1)$.

So we may assume that G admits a nowhere-zero bidirected 2-flow f such that $f(e) = 1$ for every edge $e \in E(G)$. Therefore, for any $v \in V(G)$, $E^+(v) = E^-(v)$ which implies

G is an eulerian graph. Since $\sum_{v\in V(G)} E^+(v) = \sum_{v\in V(G)} E^-(v)$ and each ordinary edge contributes 1 to both $\sum_{v\in V(G)} E^+(v)$ and $\sum_{v\in V(G)} E^-(v)$, the total contribution of non-ordinary edges to $\sum_{v\in V(G)} E^+(v)$ must be the same as that to $\sum_{v\in V(G)} E^-(v)$. Therefore, we get $|E_{\mathrm{i}}(G)| = |E_{\mathrm{o}}(G)|$ and consequently, $\mathrm{O}(G)$ is even.

"$\Longleftarrow$". Since G is an eulerian graph, there is a circuit decomposition $F = \{C_1, C_2, \ldots, C_t\}$ of $E(G)$. We will prove it by induction on t.

Suppose that $t = 1$, then G is a circuit. Since $\mathrm{O}(G)$ is even, let $\{e_1, e_2, \ldots, e_{2m}\}$ be the set of non-ordinary edges of G such that $e_1, e_2, \ldots, e_{2m}$ appear in the circuit in this order. Clearly, if we reverse the orientation of some edges in G to get another graph G', then G admits a nowhere-zero bidirected l-flow if and only if G' does.

Let us reverse the orientation of some non-ordinary edges in G so that e_i is an in-edge if i is odd and an out-edge otherwise. Because there are even number of non-ordinary edges, every two consecutive (in the circuit order) non-ordinary edges are of different types, i.e., one is an in-edge and the other one is an out-edge. Now, it is easy to reverse the orientation of some ordinary edges so that for the resulting graph G', $|E^+(v)| = |E^-(v)| = 1$ for any $v \in V(G')$. Then, G' admits a nowhere-zero bidirected 2-flow with $f(e) = 1$ for each $e \in E(G)$. It follows that G admits a nowhere-zero bidirected 2-flow.

For $t \geqslant 2$, since G is connected, then there exists C_i such that the induced subgraph $G^* = G[E(G)\backslash E(C_i)]$ is connected. Note that $\mathrm{O}(C_i)$ is even if and only if $\mathrm{O}(G^*)$ is even.

Assume that $\mathrm{O}(C_i)$ is even. By induction hypothesis, both C_i and G^* admits nowhere-zero bidirected 2-flows, therefore G admits a nowhere-zero bidirected 2-flow. So, we may assume that $\mathrm{O}(C_i)$ is odd ($\mathrm{O}(G^*)$ is odd as well). Let v be any vertex in $V(C_i) \cap V(G^*)$. Suppose $v \in C_j$ for some $j \neq i$. Let C_i' be the new circuit obtained from C_i by splitting an edge $e_i = vv_i$ away from v (becomes $e_i = v_i' v_i$, where v_i' is a new vertex) and adding a new in-edge between v and v_i' (this operation is called "expending v in C_i to an in-edge $e_i = vv_i'$). Similarly, we get a new graph G' from G^* by expending v in C_j to an out-edge $e_j = vv_j'$. Clearly, both $\mathrm{O}(C_i')$ and $\mathrm{O}(G')$ are even. By induction hypothesis, C_i' admits a nowhere-zero bidirected 2-flow f_1 and G' admits a nowhere-zero bidirected 2-flow f_2. Note that $-f_1$ is also a nowhere-zero bidirected 2-flow of C_i'. So, we can assume $f_1(e_i) = f_2(e_j)$ (otherwise since$-f_1(e_i) = f_2(e_j)$ we will use $-f_1$ instead of f_1). Therefore, let us define

$$f(e) = \begin{cases} f_1(e) & \text{if } e \in E(C_i), \\ f_2(e) & \text{if } e \in E(G)\backslash E(C_i). \end{cases}$$

Clearly, f is a nowhere-zero bidirected 2-flow of G. □

The following result is a generalization of a theorem about integer flows.

Proposition 2.1. *Let G be a bidirected graph. If G has a proper r-cycle cover, then G admits a nowhere-zero bidirected* 2^r*-flow*.

Proof. Let $F = \{P_1, P_2, \ldots, P_r\}$ be a proper r-cycle cover of G. By Proposition 1.4, each P_i admits a nowhere-zero bidirected 2-flow f_i. It is easy to verify that $\sum_{i=1}^{r} 2^{r-1} f_i$ is a nowhere-zero bidirected 2^r-flow. □

3. Proof of Theorem 1.5

Let us first introduce a useful lemma whose proof is straightforward.

Lemma 3.1. *Let G be a bidirected graph, E_0 be a subset of E and G_{E_0} be the bidirected graph obtained from G by reversing the orientation of each edge in E_0. Then we have*:

(1) *G admits a bidirected k-flow if and only if G_{E_0} admits a bidirected k-flow with the same support*.
(2) *G admits a bidirected modular k-flow if and only if G_{E_0} admits a bidirected modular k-flow with the same support*.

Proof. (1) Suppose that *f* is a bidirected *k*-flow of *G*. Let

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin E_0, \\ -f(e) & \text{if } e \in E_0. \end{cases}$$

Clearly, f' is a bidirected *k*-flow of G_{E_0} with the same support.

(2) Suppose that *f* is a bidirected modular *k*-flow of *G*. Let

$$f'(e) = \begin{cases} k - f(e) & \text{if } e \in E_0 \text{ and } f(e) \neq 0, \\ f(e) & \text{otherwise.} \end{cases}$$

Clearly, f' is a bidirected modular *k*-flow of G_{E_0} with the same support. □

Lemma 3.2. *Let G be a bidirected cubic graph. Then G admits a nowhere-zero bidirected* 3*-flow if and only if G admits a nowhere-zero bidirected modular* 3*-flow and G has a perfect matching*.

Proof. "$\Longrightarrow$". Let f be a nowhere-zero bidirected 3-flow of *G* and $E_0 = \{e \mid f(e) = -1 \text{ or } -2\}$. By the proof of Lemma 3.1(1), G_{E_0} admits a positive bidirected 3-flow f'. Clearly, f' is also a nowhere-zero bidirected modular 3-flow of G_{E_0}. By Lemma 3.1(2), *G* admits a nowhere-zero bidirected modular 3-flow. Because for every vertex $v \in V(G)$, there is exactly one incident edge *e* such that $f'(e) = 2$, then $E' = \{e \mid f'(e) = 2\}$ is a perfect matching of *G*.

"$\Longleftarrow$". Let f be a nowhere-zero bidirected modular 3-flow of *G* and $E_0 = \{e \mid f(e) = 2\}$. By Lemma 3.1(2), $G^* = G_{E_0}$ admits a nowhere-zero bidirected modular 3-flow f' such that $f'(e) = 1$ for each edge *e*. Since G^* is a cubic graph, then for any vertex v either $E(v) = E^+(v)$ or $E(v) = E^-(v)$. Let *M* be a perfect matching of G^*. Then reverse the direction of each edge *e* of *M* and change the value $f'(e)$ to be 2. The resulting nowhere-zero bidirected modular 3-flow is also a nowhere-zero bidirected 3-flow of G^*_M. By Lemma 3.1(1), G^*, and therefore *G* admit nowhere-zero bidirected 3-flows. □

Definition 3.3. Let *G* be a graph and v be a vertex of *G*. Suppose that $F \subset E(v)$, then we use $G_{[v;F]}$ for the graph obtained from *G* by splitting the edges of *F* away from v, that is, adding a new vertex v' and changing the end v of the edges in *F* to be v'.

Lemma 3.4 (*Nash-Williams [8]*). *Let k be an even integer and G be a k-edge connected graph and $v \in V(G)$. Let a be an integer such that $k \leqslant a$ and $k \leqslant d_G(v) - a$. Then there is an edge set $F \subset E(v)$ such that $|F| = a$ and $G_{[v;F]}$ is k-edge connected.*

Lemma 3.5 (*Fleischner [3]*). *Let G be a 2-edge connected graph. Suppose that v is a vertex of G with $d_G(v) \geqslant 4$ and let $e_0, e_1, e_2 \in E(v)$. Then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is 2-edge connected or $G_{[v;\{e_0,e_1,e_2\}]}$ has more components than G.*

Lemma 3.6. *Let G be a 2-edge connected graph. Suppose that v is vertex of G with $d_G(v) \geqslant 4$ and let $e_0, e_1, e_2, e_3 \in E(v)$. Then*

(1) *at least one of $G_{[v;\{e_0,e_i\}]}$ $(i = 1, 2, 3)$ is 2-edge connected*;
(2) *in the case of $d_G(v) = 4$, at least one of $G_{[v;\{e_0,e_i\}]}$ $(i = 1, 2)$ is 2-edge connected*.

Proof. By Lemma 3.5, it suffices to show that at least one of $G_{[v;\{e_0,e_1,e_2\}]}$ and $G_{[v;\{e_0,e_1,e_3\}]}$ is connected. For convenience, let $e_i = vv_i$ $(i = 0, 1, 2, 3)$. If $d_G(v) = 4$, then $G_{[v;\{e_0,e_1,e_2\}]}$ is connected. Otherwise e_3 will be an edge cut of G which is impossible. So, we may assume that $d_G(v) \geqslant 5$. Suppose that $G_{[v;\{e_0,e_1,e_2\}]}$ is disconnected. Since G is 2-edge connected then there is a path P in $G_{[v;\{e_0,e_1,e_2\}]}$ which connects v_2 to v_0 without using e_2. Therefore, $G_{[v;\{e_0,e_1,e_3\}]}$ is connected. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. "$\Longrightarrow$". Suppose that G admits a nowhere-zero bidirected 3-flow. Using the similar argument in the proof of the first part of Lemma 3.2, we can get a nowhere-zero bidirected modular 3-flow of G.

"$\Longleftarrow$". Suppose that it is not true. Let G be a smallest counterexample with respect to $|E(G)|$. By Lemma 3.1, we may assume that G admits a nowhere-zero bidirected modular 3-flow f such that $f(e) = 1$ for each edge e. Then for any vertex v with $d_G(v) = 3$ we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

Claim 1. $\delta(G) \geqslant 3$.

Otherwise, suppose that there exists $v \in V(G)$ such that $N_G(v) = \{v_1, v_2\}$. Since f is a nowhere-zero bidirected modular 3-flow of G and $f(v_1v) = f(vv_2) = 1$, we can delete the edges v_1v, vv_2 from G and add a new edge v_1v_2 (or a parallel edge if v_1v_2 exists). If $v_1v \in E^+(v_1)$ $(E^-(v_1))$, then orient this new edge such that this $v_1v_2 \in E^+(v_1)$ $(E^-(v_1))$; if $v_2v \in E^+(v_2)$ $(E^-(v_2))$, then orient this new edge such that this $v_1v_2 \in E^+(v_2)$ $(E^-(v_2))$. The resulting bidirected graph is denoted by G_0. Then G_0 is 2-edge connected and admits a nowhere-zero bidirected modular 3-flow and $|E(G_0)| < |E(G)|$. By the choice of G, G_0 admits a nowhere-zero bidirected 3-flow. Clearly, from this flow we can easily get a nowhere-zero bidirected 3-flow of G, a contradiction.

Claim 2. *There is no $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$.*

Assume that there exists a vertex $v \in V(G)$ such that $E(v) \cap E^+(v) \neq \emptyset$ and $E(v) \cap E^-(v) \neq \emptyset$. Then $d_G(v) \geqslant 4$, since, for any degree 3 vertex, either $E(v) = E^+(v)$ or $E(v) = E^-(v)$.

If $|E^-(v)|$ or $|E^+(v)| \geqslant 3$, then we may apply Lemma 3.6(1) to split two edges e', e'' away from the vertex v where $e' \in E^+(v)$ and $e'' \in E^-(v)$, and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Clearly, this new graph has a degree two vertex and if this graph admits a nowhere-zero 3-flow, then so does G. Similar to the proof of Claim 1, we can get a contradiction.

If $0 < |E^-(v)|, |E^+(v)| \leqslant 2$, then $d_G(v) = 4$ since $|E^-(v)| \equiv |E^+(v)| \pmod{3}$. By Lemma 3.6(2) and with the same argument as in Case 1, we get a contradiction as well.

By Claims 1 and 2, we have either $E(v) = E^+(v)$ or $E(v) = E^-(v)$ for any $v \in V(G)$. Therefore, $d_G(v) \equiv 0 \pmod 3$ for any $v \in V(G)$. For any v with $d_G(v) \geqslant 6$, we may apply Lemma 3.4 to split this vertex into several degree 3 vertices and the resulting graph is still 2-edge connected and f remains as a nowhere-zero bidirected modular 3-flow. Also, if the resulting graph admits a nowhere-zero bidirected 3-flow, then so does G. Recursively splitting high-degree vertices, we obtain a 2-edge connected cubic graph G^* which admits a nowhere-zero bidirected modular 3-flow.

By Petersen Theorem [9], the 2-connected, cubic graph G^* has a perfect matching. Then by Lemma 3.2, G^* admits a nowhere-zero bidirected 3-flow. So, from this flow of G^*, we can get a nowhere-zero bidirected 3-flow of G, a contradiction. □

4. Proof of Theorem 1.6

Product of flows is a method used in the proof of the theorems of 8-flow [4], 4-flow [5] and 6-flow [10]. The following lemma generalizes this method for bidirected flows of graphs and is used in the proof of Theorem 1.6.

Lemma 4.1. *Let G be a bidirected graph. If G admits a bidirected k_1-flow f_1 and a bidirected k_2-flow f_2 such that* $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$, *then G admits a nowhere-zero bidirected k_1k_2-flow.*

Proof. Let $f(e) = f_1(e) + k_1 f_2(e)$ for every $e \in E(G)$, then it is easy to verify that f is a nowhere-zero bidirected k_1k_2-flow. □

Let G be a bidirected graph. A subgraph C of G is called a *bidirected circuit* of G provided that either C is a balanced circuit of G or C is the union of two unbalanced circuits sharing exactly one vertex or C is the union of two vertex-disjoint unbalanced circuits and a simple path meeting each of the two circuits at exactly one of its end points.

We extend Seymour's closure operation [10] to bidirected graphs as follows:

For a positive integer k, if X is a subgraph of G, then the k-closure of X in G, denoted by $\langle X \rangle_k$, is the (unique) maximal subgraph of G of the form $X \cup C_1 \cup \cdots \cup C_n$, where for every i, $1 \leqslant i \leqslant n$, C_i is a bidirected circuit of G and $|E(C_i) \backslash E(X \cup C_1 \cup \cdots \cup C_{i-1})| \leqslant k$.

By [6, Theorem 4.3], we have the following:

Lemma 4.2. *Let G be a connected bidirected graph which admits a nowhere-zero bidirected flow. Let E' be a subset of $E(G)$ such that the induced graph $G[E']$ is connected and spanning. Then, $\langle E'\rangle_2 = E(G)$.*

By [6, Proposition 2.2], we have the following:

Lemma 4.3. *Let G be a bidirected graph and $k \geqslant 3$ be a prime integer. Let E' be a subset of $E(G)$ such that $\langle E'\rangle_{k-1} = E(G)$. Then G admits a bidirected integer flow f, such that, $f(e) \not\equiv 0 \,(\mathrm{mod}\, k)$ for every element e of $E(G)\backslash E'$.*

The bidirected flow *f* obtained in Lemma 4.3 can be considered as a bidirected modular *k*-flow with the same support, though the absolute value of $f(e)$ can be very large for $e \in E(G)\backslash E'$.

We will use the following lemma to get some edge-disjoint spanning trees.

Lemma 4.4 (*Nash-Williams [7], Tutte [14]*). *Every $2k$-edge-connected graph contains k-edge-disjoint spanning trees*.

Definition 4.5. Let H_1 and H_2 be two subgraphs of a graph G. The *symmetric difference* of H_1 and H_2, denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)]\backslash[E(H_1) \cap E(H_2)]$. The symmetric difference of finitely many subgraphs $\{H_1, \ldots, H_t\}$ of G is defined recursively as

$$\triangle_{1 \leqslant i \leqslant t} H_i = H_1 \triangle \cdots \triangle H_{t-1} \triangle H_t = [H_1 \triangle \cdots \triangle H_{t-1}] \triangle H_t.$$

The following is a well-known fact:

Lemma 4.6. *Let $\{H_1, \ldots, H_t\}$ be a family of subgraphs of G. Let $S = H_1 \triangle \cdots \triangle H_t$. Then*:

(1) *S is the subgraph of G induced by the edges contained in an odd number of H_i's*;
(2) *If $H_1, \ldots, H_t$ are all cycles of G, then S is also a cycle*.

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. "$\Longrightarrow$". Clearly, a nowhere-zero bidirected 6-flow is also a nowhere-zero bidirected flow.

"$\Longleftarrow$". Since G is 6-edge connected, by Lemma 4.4, G has three edge-disjoint spanning trees T_1, T_2 and T_3. Because G admits a nowhere-zero bidirected flow, by Lemma 4.2, $\langle T_i\rangle_2 = E(G)$ for $1 \leqslant i \leqslant 3$.

For $1 \leqslant i \leqslant 3$, let $k = 3$, $E' = E(T_i)$ and apply Lemma 4.3, we get a bidirected integer flow f_i of G such that $f_i(e) \not\equiv 0 \,(\mathrm{mod}\, 3)$ for every element e of $E(G)\backslash E(T_i)$. Clearly, $\mathrm{supp}(f_1) \supseteq E(G)\backslash E(T_1)$. Since $G[\mathrm{supp}(f_1)]$ contains two edge-disjoint spanning trees T_2 and T_3, the subgraph $G[\mathrm{supp}(f_1)]$ is 2-edge connected and spanning. Similarly, $\mathrm{supp}(f_2) \supseteq E(G)\backslash E(T_2)$, $\mathrm{supp}(f_3) \supseteq E(G)\backslash E(T_3)$ and both $G[\mathrm{supp}(f_2)]$ and $G[\mathrm{supp}(f_3)]$ are 2-edge connected and spanning. From each f_i $(1 \leqslant i \leqslant 3)$, we can get a bidirected modular 3-flow

f_i' such that $\text{supp}(f_i) = \text{supp}(f_i')$. Then, by Theorem 1.5, we get a bidirected 3-flow f_i'' such that $\text{supp}(f_i'') = \text{supp}(f_i)$ for $i = 1, 2, 3$.

Let T_i, T_j be two edge-disjoint spanning trees. For each $e \in T_j$, there is a unique circuit C_e of G contained in $T_i \cup \{e\}$. By Lemma 4.6(2), $C_i^j = \triangle_{e \in E(T_j)} C_e$ is a cycle. By Lemma 4.6(1), $(E(T_i) \cup E(T_j)) \supseteq E(C_i^j) \supseteq E(T_j)$, then C_i^j is also connected and spanning. Let us consider the following two cases:

Case 1: There exists one C_i^j such that $\text{O}(C_i^j)$ is even.

Without loss of generality, we may assume that $\text{O}(C_1^2)$ is even. Since C_1^2 is eulerian and connected, by Proposition 1.4, C_1^2 admits a nowhere-zero bidirected 2-flow. Therefore, G admits a bidirected 2-flow $f_{1,2}$ such that $\text{supp}(f_{1,2}) \supseteq E(T_2)$. Also f_2'' is bidirected 3-flow such that $\text{supp}(f_2'') \supseteq E(G) \backslash E(T_2)$. By Lemma 4.1, we get a nowhere-zero bidirected 6-flow of G.

Case 2: All $\text{O}(C_i^j)$'s are odd for $1 \leqslant i \neq j \leqslant 3$.

Since $\text{O}(C_1^3)$ is odd, the eulerian subgraph C_1^3 contains an unbalanced circuit c_1^3. Clearly, $E(c_1^3) \subseteq (E(T_1) \cup E(T_3))$. Because $\text{O}(C_1^2)$ is odd, $E(T_2) \subseteq E(C_1^2) \subseteq E(T_1) \cup E(T_2)$ and $E(T_2) \subseteq E(C^*)$, then $C^* = C_1^2 \triangle c_1^3$ is a connected spanning cycle of G with $\text{O}(C^*)$ even. By Proposition 1.4, C^* admits a nowhere-zero bidirected 2-flow f^* with $\text{supp}(f^*) \supseteq E(T_2)$. Similar to Case 1, by Lemma 4.1, we can get a nowhere-zero bidirected 6-flow of G. $\square$

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