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Chords of longest circuits in 3-connected graphs☆

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Abstract

Thomassen conjectured that every longest circuit of a 3-connected graph has a chord. The conjecture is verified in this paper for projective planar graphs with minimum degree at least 4.

Keywords: Longest circuit; Chord; Projective plane

1. Introduction

Thomassen [1] conjectured that every longest circuit of a 3-connected graph has a chord. In 1987, Zhang [6] proved that any longest circuit of a 3-connected planar graph G has a chord if G is cubic or $\delta \geqslant 4$. In 1997, Carsten Thomassen [4] proved that every longest circuit in 3-connected cubic graphs has a chord. In this paper, we extend a result of [6] for projective plane.

Theorem 1.1. *Let G be a 3-connected graph embedded in a projective plane with $\delta \geqslant 4$. Then every longest circuit of G must have a chord.*

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2. Terminology and notation

Throughout this paper, we consider finite simple graphs with no loops or multiple edges. For a graph $G=(V,E)$, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set, respectively. An edge e is called a chord of a circuit if e is not an edge of the circuit and both endvertices of e are on the circuit.

Let P be a subgraph of a graph G. A *P-bridge* of G is either an edge of $G\backslash E(P)$ with both ends on P or a subgraph of G induced by the edges in a component of $G\backslash V(P)$ and all edges from that component to P. For a P-bridge B of G, the vertices in $B\cap P$ are the attachments of B (on P). Define $A(B)=V(B\cap P)$ and $I(B)=V(B)-A(B)$.

Let G be a graph embedded in a surface S. A *face* of G in S is a connected component of $S-G$. The set of edges and vertices contained in *the closure* of a face F are denoted by $E(F)$ and $V(F)$, respectively.

3. Definitions and lemmas

3.1. Tutte circuits

Definition 3.1. Let G be a graph embedded in a surface and let C be a circuit of G and $\mathscr{F}=\{F_1,\ldots\}$ be a set of faces incident with C. Then the circuit C is called an *$\mathscr{F}$-Tutte circuit* if every C-bridge of G has at most three attachments and every C-bridge of G containing an edge incident with some member of $\mathscr{F}$ has at most two attachments.

Lemma 3.2 (Tutte [5], or see Ore [2]). *Let G be a 2-connected planar graph, let e_1 be an edge of G, let F_1 and F_2 be the two faces incident with e_1 and let e_2 be an edge on the boundary of F_1 and adjacent with e_1. Then there is an* $\{F_1,F_2\}$-Tutte circuit $C_{\rm T}$ *containing both e_1 and e_2.*

Lemma 3.3 (Thomas and Yu [3]). *Let G be a 2-connected graph embedded in a projective plane, let R be a face of G, and let $e\in E(R)$. Then there exists an* $\{R\}$-Tutte circuit $C_{\rm T}$ *in G such that*

(i) *$e\in E(C_{\rm T})$ and*
(ii) *every $C_{\rm T}$-bridge that contains a non-contractible circuit is edge-disjoint from R.*

3.2. Projective plane and cross caps

A projective plane $\mathscr{P}$ is a surface obtained from a closed disk by identifying points at the ends of each diameter. Note that the descriptions of a projective plane $\mathscr{P}$ are not unique. We would like to distinguish the projective plane and its representations. A representation of $\mathscr{P}$, *denoted by $\mathscr{P}_\phi$ and called a cross cap with boundary ϕ^**, is the following way of describing the surface $\mathscr{P}$ that $\mathscr{P}$ is obtained from a closed disk with boundary ϕ^* by identifying points of ϕ^* at the ends of each diagonal. After identification of diagonally opposite points, the boundary of the closed disk becomes a closed non-contractible curve in the projective plane.

Let G be a graph embedded in a projective plane $\mathscr{P}$, and $C = v_1v_2 \cdots v_rv_1$ be a non-contractible circuit of G. The circuit C is considered as a non-contractible closed curve of $\mathscr{P}$. There is an embedding of G in $\mathscr{P}$ such that the boundary of the cross cap (described in the previous paragraph) is a "circuit" $C^* = v_1' \cdots v_r'v_1'' \cdots v_r''v_1'$ where v_i' and v_i'' are the copies of the same vertex v_i of the circuit C. This is a representation of $\mathscr{P}$ as a cross cap with the boundary C^* (denoted by $\mathscr{P}_C$).

Let a graph G be embedded in a surface S and $\mathscr{F}$ be the set of all faces of G in S. Note that different embeddings of a graph G in the same surface S may have different sets of faces. So, we say two embeddings of G in the surface S are the *same* if they have *same* set of faces.

Lemma 3.4. *Let G be a graph embedded in a projective plane $\mathscr{P}$. Let ϕ be a closed non-contractible curve of $\mathscr{P}$. Then the projective plane $\mathscr{P}$ has a representation $\mathscr{P}_\phi$ with ϕ^* as its boundary of a closed disk, and the embedding of G remains the same in $\mathscr{P}$.*

Lemma 3.5. *Let a graph G be embedded in a projective plane $\mathscr{P}$. If there is a non-contractible curve ϕ of $\mathscr{P}$ such that the intersection of G and ϕ consists of at most one point, then G is a planar graph.*

Lemma 3.6. *Let G be a 3-connected graph embedded in a projective plane. Assume that G has a 3-edge-cut T and Q_1, Q_2 are the components of $G\backslash T$. For each $\{i,j\} = \{1,2\}$, let H_i be the graph obtained from G by contracting Q_j. Then one of H_1 and H_2 is planar.*

Proof. There is nothing to prove if G itself is planar. So, assume that G is not planar and has a non-contractible circuit.

(I) Assume that G has a non-contractible circuit C such that $T \cap E(C) = \emptyset$, then, without loss of generality, let Q_1 be the component of $G\backslash T$ such that $E(C) \cap Q_1 = \emptyset$. So, by Lemma 3.4, in the cross cap $\mathscr{P}_C$ (the representation of $\mathscr{P}$ with C as the boundary of the cross cap), Q_1 contains no points of the boundary of the cross cap, and therefore, H_1 is a planar graph which is embedded in the open disk $\mathscr{P}_C\backslash C$.

(II) By I, we assume that every non-contractible circuit of G must intersect both Q_1 and Q_2. Let $T = \{e_1, e_2, e_3\}$ and C be a non-contractible circuit of G.

Assume that $T \cap E(C) = \{e_1, e_2\}$ where $e_1 = v_1v_2$, $e_2 = v_iv_{i+1}$ and $v_1, v_{i+1} \in Q_1$, $v_2, v_i \in Q_2$. Without considering the cross cap, the graph G in the projective plane can be viewed as a planar graph G^* on a closed disk with C^* as the boundary of the disk (hence, v_μ' and v_μ'' are viewed as "different" vertices in cross cap $\mathscr{P}_C$, where $v_1'v_2'$, $v_1''v_2''$, $v_i'v_{i+1}'$, $v_i''v_{i+1}''$, are copies of $e_1 = v_1v_2$, $e_2 = v_iv_{i+1}$. Let $G_x^* = G^*\backslash\{v_1'v_2', v_1''v_2'', v_i'v_{i+1}', v_i''v_{i+1}'', e_3\}$ and let Q_i^* be the subgraph of G_x^* corresponding to Q_i of G ($i = 1, 2$).

(III) In the planar graph G_x^*, if there exists a $v_\mu'v_\mu''$-path P contained in some Q_μ^* for some $\mu \in \{1,2\}$, then Q_i contains a closed non-contractible circuit $v_\mu P v_\mu$ (v_μ' and v_μ'' are copies of v_μ in G). But, this contradicts to the assumption (in II) that no non-

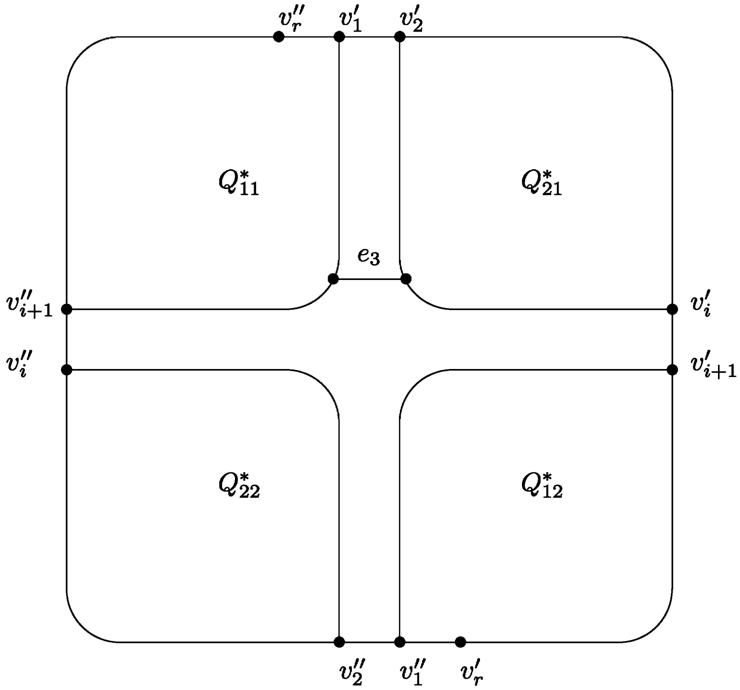

Fig. 1.

contractible circuit is completely contained in one component of $G\backslash T$. So, we assume that both Q_1^* and Q_2^* are disconnected in the planar graph G_x^*.

Let those components of Q_1^* and Q_2^* in G_x^* be Q_{11}^*, Q_{12}^* $(\subseteq Q_1^*)$, and Q_{21}^*, Q_{22}^* $(\subseteq Q_2^*)$, respectively, such that $v''_{i+1}\cdots v''_r v'_1 \subseteq Q_{11}^*$, $v'_{i+1}\cdots v'_r v''_1 \subseteq Q_{12}^*$, $v'_2\cdots v'_i \subseteq Q_{21}^*$, $v''_2\cdots v''_i \subseteq Q_{22}^*$ respectively. Without loss of generality, let $e_3 (\in T)$ be an edge joining Q_{11}^* and Q_{21}^* (see the Fig. 1).

(IV) Let ϕ be a closed non-contractible curve of the projective plane crossing through the edges $e_2 = v''_i v''_{i+1} = v'_i v'_{i+1}$, but no any other point of G since e_3 is the only edge joining Q_{11}^* and Q_{21}^* in $G_x^* \backslash \{v_1v_2, v_iv_{i+1}\}$ (see Fig. 1).

Thus, the projective plane $\mathscr{P}$ is represented as a cross cap $\mathscr{P}_\phi$ (described in the first paragraph of Subsection 3.2 and in Lemma 3.4) with ϕ^* as the boundary of the cross cap, the corresponding non-contractible curve ϕ crosses only one edge v_iv_{i+1} of G, but not others. By Lemma 3.5, G is a planar graph which contradicts the assumption that G is not planar. □

Definition 3.7. A subset S of $V(G)$ is called a *separator* of G if G has two subgraphs H_1 and H_2 such that $G = H_1 \cup H_2$ and $H_1 \cap H_2 = S$. Denote it by $[H_1, H_2]$.

Lemma 3.8. *Let G be a* 3*-connected graph embedded in a projective plane. Assume that G has a separator S with $|S| = 3$, $G = G_1 \cup G_2$ and $G_1 \cap G_2 = S$. Then one of G_i $(i = 1, 2)$ must be planar.*

Proof. Let $S=\{x_1,x_2,x_3\}=[G_1,G_2]$. Let G_1' be a graph obtained from G_1 by replacing x_i with x_i'. Let G_2' be a graph obtained from G_2 by replacing x_i with x_i''. Let G' be a new graph obtained from G_1' and G_2' by adding edges $x_i'x_i''$ $(i=1,2,3)$. Note that three edges $\{e_1,e_2,e_3\}$ is a 3-edge-cut of G'. By Lemma 3.6, one of G_1' and G_2' must be planar. Thus, one of G_1 and G_2 must be planar. □

4. Proof of Theorem 1.1

Since the same result of this theorem for planar graph was solved in [6], we pay attention to only graphs in the projective plane. Let C be a chordless longest circuit of G that satisfies the hypotheses of the theorem.

A vertex subset S is called a *separating* 3-*vertex-cut* with respect to C if S separates $V(G)$ into two sets of vertices V' and V'' such that C intersects both V' and V'' where $|S|=3$. C is *separable* if there is a *separating* 3-*vertex*-*cut* with respect to C.

I. We claim that C is separable. Assume not, that is, C is not separable.

(i) We first show that every edge of G is incident with two distinct faces. If not, then, there is a closed non-contractible curve ϕ of projective plane P that intersects with G only at one point of e, for some edge e of G. By Lemma 3.5, G is planar. Since we consider only non-planar graphs, every edge of G is incident with two distinct faces.
So we can choose the edge $e\in E(C)$ and let F be a face incident with e such that $E(F)\neq E(C)$.

(ii) For the face F, there exists an $\{F\}$-Tutte circuit C_T of G containing e by Lemma 3.3.

(iii) We claim that $V(C)\backslash V(C_T)\neq\emptyset$.

(iii(a)) *Case* 1: $E(F)=E(C_T)$. Then $E(C_T)\neq E(C)$ by (i). If $V(C)\subseteq V(C_T)$, then $V(C_T)=V(C)$ since C is a longest circuit of G. Thus, by (i), all the edges in $E(C_T)\backslash E(C)$ must be chords of C, contradicting the assumption that C is chordless.

(iii(b)) *Case* 2: $E(F)\neq E(C_T)$. Then C_T has some chords by the definition of $\{F\}$-Tutte-circuit and since G is 3-connected. If $V(C)\subseteq V(C_T)$, then $V(C_T)=V(C)$ since C is a longest circuit of G. Let $E_1=E(G[V(C)])$ and $E_2=E(G[V(C_T)])$. Here $E_1=E_2$ since $V(C)=V(C_T)$. This implies that $|E_1\backslash E(C)|$ is the same as the number of chords of C_T (which equals $|E_2\backslash E(C_T)|$). Note that the $\{F\}$-Tutte-circuit C_T does have chords. This contradicts C being chordless.

(iv) We claim that the $\{F\}$-Tutte circuit C_T has a bridge B such that $I(B)\cap V(C)\neq\emptyset$. If not, that is, every bridge B of C_T contains no vertex of C. Then $V(C)\subseteq V(C_T)$. This is in contradiction with (iii).

(v) By (iv), let B be a non-chord bridge of the $\{F\}$-Tutte circuit C_T with

$$I(B)\cap V(C)\neq\emptyset. \quad (1)$$

Since C is not separable, the 3-vertex-cut $A(B)$ (the attachments of B) cannot separate vertices of $V(C)$. Thus,

$$V(C) \subseteq A(B) \cup I(B) \tag{2}$$

and

$$V(C_T) \subseteq G \backslash I(B), \tag{3}$$

$$V(C) \cap V(C_T) \subseteq A(B). \tag{4}$$

(vi) Let $A(B) = \{x, y, z\}$. Note that the edge e is in both C and C_T. Let $e = xy$. Here, $x, y \in V(C) \cap V(C_T)$ since $e \in E(C) \cap E(C_T)$. Furthermore, we have that

$$z \in V(C) \tag{5}$$

for otherwise, $V(C) \cap V(C_T) = \{x, y\}$ and we have a circuit $xCyC_Tx$ longer than C, a contradiction.

(vii) We claim that $|V(C_T)| > 3$. If $|V(C_T)| = 3$, then, by (vi), $V(C_T) = A(B) = V(C_T) \cap V(C)$. Since $V(C) \backslash V(C_T) \neq \emptyset$ (by (iii)), $C \neq C_T$, and therefore $E(C_T) \backslash E(C) \neq \emptyset$. Hence, each edge of $E(C_T) \backslash E(C)$ is a chord of C, a contradiction.

(viii) Let Q_1 be the segment of C_T joining the vertices y and z but not containing x, and let Q_2 be the segment of C_T joining the vertices z and x but not containing y.
By (5), $z \in V(C)$ and C_T is of length $\geqslant 4$ by (vii). Therefore, either the segment $Q_1 = yC_Tz$ or the segment $Q_2 = zC_Tx$ has at least one vertex v that is not in C. Say, $v \in Q_2 \backslash V(C)$. Since G is 3-connected, there is a path P joining $Q_2 \backslash \{x, z\}$ and $Q_1 \backslash \{z\}$ in $G \backslash \{x, z\}$. Choose P to be internally disjoint from $Q_2 \cup Q_1$ and as short as possible. Let $P = u' \cdots u''$ with $u' \in Q_2 \backslash \{x, z\}$ and $u'' \in Q_1 \backslash \{z\}$. By the choice of P, we have that $[V(P) \backslash \{y\}] \cap I(B) = \emptyset$, then $u'Q_2xCyQ_1u''Pu'$ is a circuit longer than C, a contradiction.

II. By I, the longest circuit C is separable. Let V^* be a separable 3-vertex-cut with respect to C such that V^* separates G into V' and V''. By Lemma 3.6 one of $G[V' \cup V^*]$, $G[V'' \cup V^*]$ must be planar. Without loss of generality, assume that $G[V'' \cup V^*]$ is a planar. We choose V^* such that V'' is as small as possible.

Since C must pass through two vertices of V^* to enter V'' from V', the circuit C is the union of two paths $P' = x \cdots y$ and $P'' = y \cdots x$ contained in $G[V' \cup V^*]$ and $G[V'' \cup V^*]$, respectively, and $x, y \in V^*$. Let $V^* = \{x, y, z\}$. We construct a new graph G^* according to the following two cases:

(a) If $z \notin V(P')$, let w be a new vertex not in G. Define G^* to be the graph obtained from $G[V'' \cup V^*]$ by adding a vertex w and three new edges wx, wy and wz.
(b) If $z \in V(P')$, let $w = z$. Define G^* to be the graph obtained from $G[V'' \cup V^*]$ by adding two new edges wx and wy.

Obviously, G^* is still planar.

Let C^* be the circuit obtained from the path P'' by adding the vertex w and edges wx, wy. Since $P'' = y \cdots x = C \cap G[V'' \cup V^*]$ and C is longest circuit in G, hence, C^* is a longest circuit of G^* containing wx and wy.

Let F_1, F_2 be two faces of G containing wx. There is an $\{F_1, F_2\}$-Tutte circuit C^o of G^* that contains wx and wy (by Lemma 3.2).

(i) If $|V(C) \cap V''| = 1$, then let $\{v\} = V(C) \cap V''$ and P'' must be one of $\{xvy, xvzy, xzvy\}$. Without loss of generality, let $P'' = w_1 \cdots w_t$ where $w_1 = x$, $w_t = y$, $w_2 = v$ if $t = 3$ and $w_3 = z$ if $t = 4$, (note $xzvy$ is similar to $xvzy$).
Since the minimum degree of G is at least four, there are at least two edges adjacent to v not contained in C. Let vu be an edge not contained in C such that $z \neq u$. Clearly $u \in V''$. Let B_C be the C-bridge of G containing u. If w_1 or w_3 is an attachment of B_C, then $P^* = w_1 P_{w_1} uvP'' w_t$ (where P_{w_1} is the longest path in B_C connecting w_1 and u) or $P^* = xvuP_{w_3} w_3 P'' w_t$ (where P_{w_3} is the longest path in B_C connecting w_3 and u) would be longer than P'' in G^*, C would not be a longest circuit. Hence, neither w_1 nor w_3 is in attachment set of B_C. Note that every path joining u and a vertex in $V(C)$ must pass through $\{w_4, v\}$ if $t = 4$ or $\{z, v\}$ if $t = 3$, which contradicts G being 3-connected. Hence, $|V(C) \cap V''| \geqslant 2$.

(ii) Since V'' is minimum and $|V(C) \cap V''| \geqslant 2$, so $|N(v) \cap V''| \geqslant 2$ for any $v \in V^*$. Let xx_i be the edge of G^* lying on the boundary of F_i and $xx_i \neq xw$ for $i = 1, 2$. Obviously, $xx_1, xx_2 \in E(G)$. Since C^o is an $\{F_1, F_2\}$-Tutte circuit and because G^* is 3-connected, xx_i either lies on C^o or is a chord of C^o. Hence, $\{x, x_1, x_2\} \subseteq V(C^o)$ and one of $\{xx_1, xx_2\}$ must be a chord of C^o since $xw \in E(C^o)$.

(iii) We claim that $V(C^o) \setminus \{x, y, z, w\} \neq \emptyset$. Assume that $V(C^o) \setminus \{x, y, z, w\} = \emptyset$, then by (ii), $\{x_1, x_2\} = \{y, z\}$ and since $V(C) \cap V'' \neq \emptyset$, xy is a chord of C. This contradicts C being chordless. Hence, $V(C^o) \setminus \{x, y, z, w\} \neq \emptyset$.

(iv) We claim that each non-chord bridge B of C^o must be contained in some bridge of C^*. Suppose that $I(B) \cap V(C^*) \neq \emptyset$ for some bridge B of C^o.

(iv α) *Case* 1: $w \notin A(B)$ or $w = z$. If $w \notin A(B)$, then $z \notin I(B)$ since C^o contains edges wx and wy. If $w = z$, then $w = z \notin I(B)$. Hence $x, y, z \notin I(B)$ and V' adjacent only with $\{x, y, z\}$ in G will imply that $A(B)$ is a vertex-cut that separates G into $I(B)$ and $V(G) \setminus [A(B) \cup I(B)]$. Since $V(C^o) \setminus \{x, y, z, w\} \neq \emptyset$ (by (iii)), $I(B)$ would be a proper subset of V''. However, $I(B)$ intersects with C, which contradicts choice of V^* with V'' minimal.

(iv β) *Case* 2: $w \in A(B)$ and $w \neq z$. Since $\{x, y, w\} \subseteq V(C^o)$ and $w \in A(B)$, we must have that $z \in I(B)$ and $wz \in [I(B), A(B)]$. Since C^o is an $\{F_1, F_2\}$-Tutte circuit and $wz \in E(F_1 \cup F_2)$, B is a 2-attachment bridge.
Since $d(z) \geqslant 3$ in $G^*, I(B) \setminus \{z\} \neq \emptyset$. Let $A(B) = \{w, u\}$. Then $U^* = \{z, u\}$ is a vertex-cut of G^* which separates G^* into $U'' = I(B) \setminus \{z\}$ and $U' = [V(G^*) \setminus [I(B) \cup A(B))] \cup \{w\}$. Since $\{x, y, z, w\} \subseteq U' \cup U^*$, V' only adjacent with $V^* = \{x, y, z\}$ would imply that V' and U'' are disconnected in $G \setminus U^*$. Hence, U^* is a 2-vertex-cut separating G into U'' and $V' \cup U' \setminus \{w\}$, which contradicts the assumption that G is 3-connected. Now we conclude our claim in all cases.

(v) By (iv), $V(C^*) \subseteq V(C^o)$ since each bridge of C^o is contained in some bridge of C^*. Moreover, $V(C^o) = V(C^*)$ because C^* is a longest circuit of G^* containing wx and wy. By (ii), $\{x, x_1, x_2\} \subseteq V(C^o) = V(C^*)$ and one of $\{xx_1, xx_2\}$ is a chord of C^*, which is also a chord of C in G. This contradicts C being chordless. This completes the proof. □

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