ELSEVIER

Discrete Mathematics 154 (1996) 245–253

DISCRETE MATHEMATICS

Nowhere-zero 4-flows and cycle double covers

Cun-Quan Zhang [1]

Department of Mathematics, West Virginia University, Morgantown, WV, 26506-6310, USA

Received 10 September 1993; revised 27 September 1994

Abstract

In this paper, we obtained some necessary and sufficient conditions for a graph having 5-, 6- and 7-cycle double covers, etc. We also provide a few necessary and sufficient conditions for a graph admitting a nowhere-zero 4-flow. With the aid of those basic properties of nowhere-zero 4-flow and the result about 5-cycle double cover, we are able to prove that each 2-edge-connected graph with one edge short of admitting a nowhere-zero 4-flow has a 5-cycle double cover which is a generalization of a theorem due to Huck and Kochol (JCTB, 1995) for cubic graphs.

1. Introduction

All graphs we consider in this paper are all 2-edge-connected. By a circuit, we mean a connected 2-regular graph. By a cycle, we mean a graph with the degree of every vertex being even. (Thus, a cycle is a union of edge-disjoint circuits and some isolated vertices.) Let F be a subset of $E(G)$. The graph obtained from G by contracting all edges of F is denoted by G/F. Let H be a subgraph of G and $v \in V(G)$. The degree of v in H is denoted by $d_H(v)$. All other standard graph-theoretic terms that are used in this paper can be found for instance in [4].

1.1. Integer flows

Let $G = (V,E)$ be a graph with vertex set V and edge set E. Let (D, f) be an ordered pair where D is an orientation of $E(G)$ and f is a weight on $E(G): E(G) \mapsto Z$, where Z is the set of all integers. For each $v \in V(G)$, denote

$$f^+(v) = \sum\{f(e)\} \quad (\text{or, } f^-(v) = \sum\{f(e)\}),$$

[1] Partial support provided by the National Science Foundation under Grant DMS-9104824.

SSDI 0012-365X(95)00047-X

where the summation is taken over all oriented edges of G (under the orientation D) with tails (or, heads, respectively) at the vertex v.

Definition 1.1. (1) An *integer flow* of G is an ordered pair (D, f) such that

$$f^+(v) = f^-(v)$$

for every vertex $v \in V(G)$.

(2) A *k-flow* of G is an integer flow (D, f) such that $|f(e)| < k$ for every edge of G.

(3) The *support* of a weight f is the set of all edges of G with $f(e) \neq 0$ and is denoted by $\text{supp}(f)$.

(4) A *nowhere-zero k-flow* (D, f) of a graph G is a k-flow such that $\text{supp}(f) = E(G)$.

The concept of integer-flow was introduced by Tutte ([21] also see [23,12], etc.) as a refinement and generalization of the face-coloring and edge-3-coloring problems. The following conjectures are the fundamental problems in this area.

Conjecture 1.1 (Tutte [22]). Every 2-edge-connected graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

Conjecture 1.2 (Tutte [21]). Every 2-edge-connected graph admits a nowhere-zero 5-flow.

Conjecture 1.3 (Tutte, see [17] and [18]). Every 4-edge-connected graph admits a nowhere-zero 3-flow.

The topic of integer flow is one of the most important research topics in graph theory. Many articles have been published in this area (see surveys [12,13,23], etc.) In this paper, we are mainly interested in the topic of nowhere-zero 4-flow since it is very closely related to the topic of cycle cover problem. (One of the major results in this topic is a theorem of Jaeger [10] that every 4-edge-connected graph admits a nowhere-zero 4-flow.) As a generalization of edge-3-colorings of cubic graphs, graphs admitting nowhere-zero 4-flows have many properties that correspond to properties of edge-3-colorable cubic graph. In Section 2, we will provide some direct proofs of these properties without converting graphs into cubic graphs by a traditional vertex-splitting method since the method is usually tedious and does not lead to a minor-closure result.

1.2. Cycle covers

Definition 1.2 (1) A family $\mathscr{F}$ of cycles of a graph G is called a *cycle cover* if each edge of G is contained in a cycle of $\mathscr{F}$.

(2) A cycle cover $\mathscr{F}$ of G is called a *cycle double cover* of G if each edge of G is contained in precisely two cycles of $\mathscr{F}$.

(3) A family $\mathscr{F}$ of cycles of a graph G is called a *k-cycle double cover* of G if $\mathscr{F}$ is a cycle double cover of G consisting of at most k cycles of G.

The following is a very famous conjecture in graph theory. Many papers about this conjecture have been published in recent years. (See surveys [9,11,26], etc.)

Conjecture 1.4 (Szekeres [19] and Seymour [16]. Or see survey [11]). Every 2-edge-connected graph has a cycle double cover.

Motivated by the 5-flow conjecture (Conjecture 1.2), a stronger conjecture was proposed by Preissmann and Celmins.

Conjecture 1.5 (Preissmann [15] and Celmins [5]). Every 2-edge-connected graph has a 5-cycle double cover.

In this paper, we will obtain some necessary and sufficient conditions for graphs having a 5-cycle double cover (and 6-cycle double cover, 7-cycle double cover, ..., as well). With these results and some basic properties in the theory of integer flow, we generalize a result of Huck and Kochol [8] that every 2-edge-connected graph with at most one edge short of admitting a nowhere-zero 4-flow has a 5-cycle double cover.

2. Nowhere-zero 4-flow

Definition 2.1. Let H_1 and H_2 be two subgraphs of a graph G, the *symmetric difference* of H_1 and H_2, denoted by $H_1 \triangle H_2$, is the subgraph of G induced by the set of edges $[E(H_1) \cup E(H_2)] \setminus [E(H_1) \cap E(H_2)]$.

The following facts are either straightforward or elementary:

1. Let H_1 and H_2 be two subgraphs of G and $S = H_1 \triangle H_2$. Then for each vertex v of G

$$d_S(v) \equiv d_{H_1}(v) + d_{H_2}(v) \pmod 2,$$

2. If C_1, C_2 are cycles of a graph G, then $C_1 \triangle C_2$ is also a cycle of G.
3. Let $\mathscr{F}$ be a cycle cover of a graph G. Then the subgraph of G induced by the edges contained in an odd number of cycles of $\mathscr{F}$ induces a cycle of G.
4. If (D, f) is a 2-flow of G then supp(f) is a cycle of G.
5. If (D, f) is a k-flow of G then the set of edges with odd weights is a cycle of G.

We present a few basic properties and equivalent definitions of nowhere-zero 4-flow, some of which are well-known by many mathematicians in this area.

2.1. Cycle double cover

Theorem 2.1 (Jaeger [11]). *Let $G = (V, E)$ be a graph. The following statements are equivalent:*

(i) *G admits a nowhere-zero 4-flow,*
(ii) *G has a 3-cycle double cover,*
(iii) *G has a 4-cycle double cover.*

We are not to present the proof of this theorem, since it has appeared in many papers already.

2.2. Parity subgraphs

The following notion we give here was introduced by Celmins in [5].

Definition 2.2. (1) A spanning subgraph H of a graph G is called a *parity subgraph* of G if for each vertex $v \in V(G)$

$$d_H(v) \equiv d_G(v) \pmod 2$$

(2) A decomposition of the edge-set of a graph G is called *parity subgraph decomposition* if each part of the decomposition is a parity subgraph of G.

(3) A parity subgraph decomposition of G is *trivial* if it has only one element (the graph G itself).

The following are some observations.

1. A graph G is a parity subgraph of itself and every graph has a parity subgraph decomposition.
2. A subgraph H of G is a parity subgraph if and only if $G \backslash E(H)$ is a cycle of G.
3. The union of an even (odd) number of edge-disjoint parity subgraphs is a cycle (parity subgraph).
4. The number of parity subgraphs in a parity subgraph decomposition of a graph is always odd.

Theorem 2.2. *A graph G admits a nowhere-zero 4-flow if and only if G has a non-trivial parity subgraph decomposition.*

Proof. By Theorem 2.1, we only need to prove that G has a 3-cycle double cover if and only if G has a parity subgraph decomposition consisting of three parity subgraphs. (Note that if a non-trivial parity subgraph decomposition $\mathscr{P}$ of G has $t \geqslant 3$ (t odd) elements then replacing $t-2$ elements of $\mathscr{P}$ by their union yields a non-trivial parity subgraph decomposition with precisely three elements). Thus the theorem is obvious since

(i) for each cycle double cover $\mathscr{F}$ of G,

$$\{G \backslash E(C) \colon\ C \in \mathscr{F}\}$$

is a non-trivial parity subgraph decomposition if and only if $|\mathscr{F}| = 3$; and

(ii) for each non-trivial parity subgraph decomposition $\mathscr{P}$ of G,

$$\{G\backslash E(P):\ P \in \mathscr{P}\}$$

is a cycle double cover of G if and only if $|\mathscr{P}| = 3$. □

A non-trivial parity subgraph decomposition of a graph G is a generalization of edge-3-colorings of cubic graphs since a non-trivial parity subgraph decomposition of a cubic graph is a set of three edge-disjoint perfect matchings.

2.3. Evenly spanning cycles

Definition 2.3. (1) A vertex of a graph G is *odd* (or *even*) if the degree of the vertex is odd (or even, respectively).

(2) If C is a cycle of a graph G, a component N of C is *odd* (or, *even*) if N contains odd (or, even, respectively) number of odd vertices of G.

(3) A cycle C of G is *evenly spanning* if C contains all odd vertices of G and each component of C is even.

(The topic of evenly spanning cycle was studied in [3] for graphs with 4-colorable embedding on orientable surfaces.)

Lemma 2.3. *A cycle S is evenly spanning in G if and only if S is the union of two edge-disjoint parity subgraphs of G.*

Proof. (i) The proof of the "if" part. Let P_1, P_2 be two edge-disjoint parity subgraphs of G such that $S = P_1 \cup P_2$. It is obvious that $S = P_1 \cup P_2$ is a cycle of G. Since each component of P_1 contains an even number of odd vertices of P_1 and P_1 is a parity subgraph of G, each component of P_1 contains an even number of odd vertices of G. Since the vertex set of each component C of S is the union of the vertex sets of several components of P_1, the component C of S also contains an even number of odd vertices of G. All these prove that $S = P_1 \cup P_2$ is an evenly spanning cycle of G.

(ii) The proof of the "only if" part. Let S be an evenly spanning cycle of G and let $V_o = \{v_1, \ldots, v_{2t}\}$ be the set of all odd vertices of G such that v_{2i-1} and v_{2i} are contained in the same component of S for each $i = 1, \ldots, t$. Let P_i be a path joining v_{2i-1}, v_{2i} and contained in S. Noting that the symmetric difference Q of all P_i's $(i = 1, \ldots, t)$ is a parity subgraph of G. Thus, Q and $S\backslash E(Q)$ are two edge-disjoint parity subgraph of G whose union is S. □

With the fact that each parity subgraph decomposition of a graph always contains an odd number of elements, (by observation (4)), we have the following theorem which is an immediate corollary of Lemma 2.3 and Theorem 2.2.

Theorem 2.4. *A graph G admits a nowhere-zero 4-flow if and only if G has an evenly spanning cycle.*

It is evident that Theorem 2.4 is a generalization of the following fact:

A cubic graph G is edge-3-colorable if and only if G has a 2-factor C such that each component of C is a circuit of even length.

2.4. Faithful cycle covers

The following theorem for the cubic case was used by Seymour in [16] and is generalized for the general case in [24].

Theorem 2.5 (Zhang [24]). *Let G be a graph. The graph G admits a nowhere-zero 4-flow if and only if for each cycle C of G, G has a 4-cycle double cover $\mathscr{C}$ such that $C \in \mathscr{C}$.*

Proof. By Theorem 2.1, we only need to prove the "only if" part of the theorem and let $\{C_1, C_2, C_3\}$ be a 3-cycle double cover of G. Then $\mathscr{F} = \{C, C\triangle C_1, C\triangle C_2, C\triangle C_3\}$ is a 4-cycle double cover of G containing the given cycle C. □

In the case of the cycle double cover (described in Theorem 2.5) containing at most three cycles, we have the following structural result, which will be useful in the later discussions.

Theorem 2.6. *Let C be a cycle of G. Then the graph G has a 3-cycle double cover $\mathscr{F}$ containing C if and only if C is an evenly spanning cycle of G.*

Proof. (i) The proof of the "if" part: By Lemma 2.3, let $\mathscr{P} = \{P_1, P_2, P_3\}$ be a parity subgraph decomposition of G such that $P_1 = G \setminus E(C)$, and $C = P_2 \cup P_3$. Then, obviously, $\{P_1 \cup P_2, P_1 \cup P_3, C\}$ is a 3-cycle double cover of G.

(ii) The proof of the "only if" part: Let $\{C_1, C_2, C\}$ be a 3-cycle double cover of G. Since $C = C_1 \triangle C_2$, the cycle C is the union of two parity subgraphs $\{G \setminus E(C_1), G \setminus E(C_2)\}$. By Lemma 2.3, C is an evenly spanning cycle of G. □

The concept of a cycle double cover containing a given cycle is equivalent to the concept of faithful cycle cover of a $(1,2)$-eulerian weight (see [1,2,6,9,16,24]).

3. 5-cycle double covers

Theorem 3.1. *A graph G has a 5-cycle double cover if and only if G has two subgraphs A and B such that*

(i) $E(G) = E(A) \cup E(B)$,

(ii) *$A \cap B = C$ is a cycle of G,*
(iii) *each A and B admits a nowhere-zero 4-flow, and*
(iv) *C is an evenly spanning cycle of A*

Proof. (i) The proof of the "if" part of the theorem. By Theorems 2.5, and the subgraph A has a 3-cycle double cover $\mathscr{C}_1 = \{C, C_1, C_2\}$ and B has a 4-cycle double cover $\mathscr{C}_2 = \{C, C_3, C_4, C_5\}$. Thus, $\{C_1, \ldots, C_5\}$ is a 5-cycle double cover of G.

(ii) The proof of the "only if" part of the theorem. Let $\{C_1, \ldots, C_5\}$ be a 5-cycle double cover of a graph G. Let A be the subgraph of G induced by the edges contained in C_1, C_2, and let B be the subgraph of G induced by the edges contained in C_3, C_4, C_5. Since the edge set of the intersection of A and B consists of edges of A contained in only one cycle of the cycle cover $\{C_1, C_2\}$ of the subgraph A, we have that $A \cap B = C$ is a cycle of G. Thus, A has a 3-cycle double covers $\{C, C_1, C_2\}$ and B has a 4-cycle double cover $\{C, C_3, C_4, C_5\}$. By Theorem 2.1, both A and B admit nowhere-zero 4-flows. By Theorem 2.6, C is an evenly spanning cycle of A. $\square$

With similar proofs as above (without applying Theorem 2.6), we have the following relaxed theorems.

Theorem 3.2. *A graph G has a 6-cycle double cover if and only if G has two subgraphs A and B such that*
(i) *$E(G) = E(A) \cup E(B)$,*
(ii) *$A \cap B = C$ is a cycle of G, and*
(iii) *each A and B admits a nowhere-zero 4-flow.*

Theorem 3.3. *A graph G has a 7-cycle double cover if and only if G has two subgraphs A, B and two cycles C, D such that*
(i) *$E(G) = E(A) \cup E(B)$,*
(ii) *$C \subseteq A$ and $D \subseteq B$,*
(iii) *$A \cap B \subseteq C$ and $A \cap B \subseteq D$, and*
(iv) *each A and B admits a nowhere-zero 4-flow.*

The following conjecture is proposed as an approach for verifying Conjecture 1.5 (as well as the famous Cycle Double Cover Conjecture).

Conjecture 3.4. Let S be a spanning cycle of a 3-edge-connected graph G with the least number of odd components. Let Q be a smallest, bridgeless, parity subgraph of $G/E(S)$. Then the subgraph of G induced by all edges of S and all edges of Q admits a nowhere-zero 4-flow.

Let A be the subgraph of G induced by $E(G) \setminus E(Q)$ and B be the subgraph of G induced by $E(S) \cup E(Q)$. Here S is an evenly spanning cycle of A. The number of edges of $E(G) \setminus E(Q)$ linking a component C of S and $A \setminus V(C)$ is even. It implies that C contains an even number of odd degree vertices of A. Hence S is an evenly spanning cycle of A. By Theorem 3.1, Conjecture 1.5 follows.

For a graph, which is "almost" admitting a nowhere-zero 4-flow, Conjecture 3.4 is true and therefore Conjecture 1.5 is true. (Note, the concept of "almost admitting a nowhere-zero 4-flow" was introduced by Jaeger in [12]. A graph with this property is called a 'nearly' F_4 graph in [12].)

The following theorem was originally discovered by Huck and Kochol [8] independently.

Theorem 3.5 (Huck and Kochol [8]). *Let G be an edge-3-colorable cubic graph and e be an edge of G such that $G' = G\backslash\{e\}$ is 2-edge-connected. Then G' has a 5-cycle double cover.*

The proof of Theorem 3.5 in [8] can be adapted and modified for proving the following lemma.

Lemma 3.6. *Let S be a spanning cycle of a 3-edge-connected graph G. Let Q be a smallest, bridge-less, parity subgraph of $G/E(S)$. If S has at most two odd components, then the subgraph of G induced by all edges of S and all edges of Q admits a nowhere-zero 4-flow.*

By Theorem 3.1 and Lemma 3.6, we obtain an immediate corollary which is a generalization of Theorem 3.5 for graphs without the restriction of 3-regularity.

Theorem 3.7. *Let G be a graph admitting a nowhere-zero 4-flow and e be an edge of G such that $G' = G\backslash\{e\}$ is 2-edge-connected. Then G' has a 5-cycle double cover.*

An important application of Theorem 3.5 (as well as Theorem 3.7) is the following result.

Theorem 3.8 (Huck and Kochol [8]). *Every 2-edge-connected graph containing a Hamilton path has a 5-cycle double cover.*

Theorem 3.8 was originally discovered by Tarsi ([20] and also see [7] for a simplified proof) for a 6-cycle double cover and recently improved by Kochol and Huck independently (which will be published jointly [8]).

References

[1] B. Alspach, L. Goddyn and C.Q. Zhang, Graphs with the circuit cover property, Trans. AMS 344 (1994) 131–154.
[2] B. Alspach and C.Q. Zhang, Cycle coverings of cubic multigraphs, Discrete Math. 111 (1993) 11–17.
[3] D. Archdeacon, Face colorings of embedded graphs, J. Graph Theory 8 (1984) 387–398.
[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London, 1976).

[5] U.A. Celmins, On cubic graphs that do not have an edge 3-coloring, Ph.D. Thesis, Univ. of Waterloo, Waterloo, Canada, 1984.
[6] H. Fleischner and A. Frank, On cycle decomposition of eulerian graph, J. Combin. Theory B 50 (1990) 245–253.
[7] L. Goddyn, Cycle double covers of graphs with Hamilton paths, J. Combin. Theory B 46 (1989) 253–254.
[8] A. Huck and M. Kochol, Five cycle double covers of some cubic graphs, J. Combin. Theory B 64 (1995) 119–125.
[9] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, British Combinatorics Conf. preprint, 1993.
[10] F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory B 26 (1979) 205–216.
[11] F. Jaeger, A survey of the cycle double cover conjecture, in: B. Alspach and C. Godsil, eds., Cycles in Graphs; Ann. Discrete Math. 27 (1985) 1–12.
[12] F. Jaeger, Nowhere-zero flow problems, in: L. Beineke and R. Wilson, eds., Selected Topics in Graph Theory 3 (Wiley, New York, 1988) 71–95.
[13] T. Jensen and B. Toft, Graph Colorings Problems (Wiley, New York, 1994).
[14] H-J. Lai and C.-Q. Zhang, Nowhere-zero 3-flows of highly connected graphs, Discrete Math. 110 (1992) 179–183.
[15] M. Preissmann, Sur les colorations des arêtes des graphes cubiques, Thèse de Doctorat de 3^{eme}, Grenoble, 1981.
[16] P.D. Seymour, Sums of circuits, in: J. A. Bondy and U. S. R. Murty, eds., Graph Theory and Related Topics (Academic Press, New York, 1979) 342–355.
[17] R. Steinberg, Grötzsch's theorem dualized, M. Math Thesis, Univ. of Waterloo, Ontario, Canada, 1976.
[18] R. Steinberg and D. H. Younger, Grötzsch's theorem for the projective plane, Ars Combin. 28 (1989) 15–31.
[19] G. Szekeres, Polyhedral decompositions of cubic graphs, J. Austral. Math. Soc. 8 (1973) 367–387.
[20] M. Tarsi, Semi-duality and the cycle double cover conjecture, J. Combin. Theory B 41 (1986) 332–340.
[21] W. Tutte, A contribution on the theory of chromatic polynomial, Can. J. Math. 6 (1954) 80–91.
[22] W.T. Tutte, On the algebraic theory of graph colourings, J. Combin. Theory 1 (1966) 15–50.
[23] D.H. Younger, Integer flows, J. Graph Theory 7 (1983) 349–357.
[24] C.-Q. Zhang, Minimum cycle coverings and integer flows, J. Graph Theory 14 (1990) 537–546.
[25] C.-Q. Zhang, Cycle covers and cycle decompositions of graphs, Ann. Discrete Math. 55 (1993) 183–190.
[26] C.-Q. Zhang, Cycle cover theorems and their applications. in: N. Robertson and P. Seymour eds., Graph Structure Theory Contemporary Math. 147 (1993) 405–418.