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Cycle covers of cubic multigraphs

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Abstract

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1. Introduction

The terms graph and multigraph will be used throughout this paper. A *graph* has neither multiple edges nor loops, while a *multigraph* may have multiple edges but no loops.

A *weight* w on a multigraph G is a function mapping the edge-set $E(G)$ to real numbers. If e is an edge of G, $w(e)$ is said to be the *weight* of the edge e. Throughout this paper, we consider only weights w for which $w(e)$ is always a nonnegative integer. In particular, a weight w on G is called a *(1, 2)-weight* if $w(e) \in \{1, 2\}$ for every $e \in E(G)$.

A *weighted multigraph* is a pair (G, w), where G is a multigraph and w is a weight on G. If A is a set of edges of G, then the *weight* of A is denoted as $w(A)$ and defined by

$$w(A) = \sum_{e \in A} w(e). \tag{1}$$

We use $G \backslash A$ to denote the submultigraph obtained by removing the edges of A from G. For simplicity, $G \backslash e$ will be used rather than $G \backslash \{e\}$. The same notation will be used for deleting a set of vertices. If every edge-cut of G has even weight, then w is said to be an *eulerian weight*. The weight w is said to be *balanced* if, for every edge-cut A of G and

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for every $e \in A$, $w(e) \leqslant w(A-e)$. If the weight w is both balanced and eulerian, we shall say w is *admissible*. In particular, if G is a 2-edge-connected multigraph, a $(1,2)$-weight is admissible if and only if it is eulerian, that is, if and only if the edges with weight 1 form an edge-disjoint union of cycles.

Given a weighted multigraph (G, w), a family $\boldsymbol{F}$ of cycles such that each edge e of G is contained in precisely $w(e)$ cycles of $\boldsymbol{F}$ is called a *cycle w-cover* of G. Clearly, if G has a cycle w-cover, w is admissible. It is important to note that 2-cycles are allowed but the two edges in a 2-cycle must be different edges. Not every weighted multigraph with an admissible weight w has a cycle w-cover. For example, if the edges of a 1-factor of the Petersen graph P are given weight 2 and the remaining edges are given weight 1, the resulting weight w is admissible, but there is no cycle w-cover of (P, w) [7, 9]. In what follows, the preceding example plays a major role and will be denoted as P^* throughout the paper.

Seymour [9] proved that any planar multigraph has a cycle w-cover for any admissible weight w. His proof uses the four-color theorem. The main theorem to follow provides a proof of Seymour's result without employing the four-color theorem. Such proofs already exist [5, 6], but our result also provides an extension to admissible $(1,2)$-weighted cubic multigraphs not containing a subdivision of the Petersen graph. In addition, applications concerning the cycle double-cover conjecture, and the equivalence of the Chinese postman problem and the minimum cycle covering problem are considered in Section 3.

2. Main results

The following result has been observed by many people. We include its proof because it is short.

Lemma 2.1. *Every 3-edge-colorable cubic multigraph has a cycle w-cover for any admissible* $(1,2)$*-weight w.*

Proof. Let w be an admissible $(1,2)$-weight defined on a 3-edge-colorable cubic multigraph G. Let the edge-set of G be colored by colors 1, 2 and 3. Let

$$E_{i,j} = \{e \in E(G)\colon e \text{ is colored either } i \text{ or } j\}$$

and

$$E^t = \{e \in E(G)\colon w(e) = t\}.$$

Let $C_{i,j}$ be the set of cycles making up the subgraph $E_{i,j} \Delta E^1$, where Δ denotes the symmetric difference of two sets. It is easy to verify that $\boldsymbol{F} = C_{1,2} \cup C_{1,3} \cup C_{2,3}$ is a cycle w-cover of G. $\square$

Let (G, w) be a multigraph with a $(1,2)$-eulerian weight w and V_2 be the set of vertices of degree 2 in G. Let $x \in V_2$ and $xu, xv \in E(G)$. We define an operation on (G, w)

by deleting x, xu, and xv, and adding a new edge uv to $G \setminus x$ with weight equal to $w(xu)$. Since w is eulerian, $w(xu) = w(xv)$ and the operation makes sense, although it may take us out of the class of multigraphs if loops arise. We shall see that loops cannot arise in our context. This operation is denoted as Π_x and the new $(1,2)$-eulerian weighted multigraph is denoted as $\Pi_x(G,w) = (\Pi_x(G), \Pi_x(w))$. Obviously, G is a subdivision of $\Pi_x(G)$. We can view $\Pi_x(w)$ as identical to w and shall write w for $\Pi_x(w)$. Let $\Pi(G,w) = \Pi_{x_1}(w)\Pi_{x_2} \cdots \Pi_{x_t}(G,w)$, where $V_2 = \{x_1, x_2, \ldots, x_t\}$.

Definition 2.2. An eulerian weighted multigraph (H, w') is said to be *smaller* than another eulerian weighted multigraph (G, w), denoted as $(H, w') \propto (G, w)$, if either

(i) there exists an isomorphism f from H to G such that $w'(e') \leqslant w(f(e'))$ for each $e' \in E(H)$, or

(ii) G and H are not isomorphic but G has a submultigraph H' isomorphic to a subdivision of H.

Additionally, we say (H, w') is *properly smaller* if either (i) holds and $w' \neq w$ or (ii) holds.

It is easy to see that the relation $\propto$ defined above is a partial order on the class of eulerian weighted multigraphs.

If G is an eulerian $(1,2)$-weighted cubic multigraph, H is a 2-edge-connected proper subgraph of G and H is not a cycle, then $\Pi(H)$ is a cubic multigraph smaller than G. The proof of the following result is obvious.

Proposition 2.3. *The eulerian-$(1,2)$-weighted multigraph (H, w) has a cycle w-cover if and only if $\Pi(H, w)$ has a cycle w-cover.*

Definition 2.4. Let $C_1 = u_1 u_2 \cdots u_r u_1$ and $C_2 = v_1 v_2 \cdots v_r v_1$ be two vertex-disjoint copies of the r-cycle, $r \geqslant 3$, and let σ be any permutation of $\{1, 2, \ldots, r\}$. The cubic graph obtained by joining u_i with $v_{\sigma(i)}$, $i = 1, 2, \ldots, r$, is called the *σ-prism* over the r-cycle. Note that the Petersen graph is a σ-prism over the 5-cycle, where $\sigma = (1)(2\,4\,5\,3)$ is one possible permutation that works to show this.

The following result was proved by Ellingham [3]. It is used in the proof of Theorem 2.6.

Theorem 2.5. *Every non-3-edge-colorable σ-prism over a cycle contains a subdivision of the Petersen graph.*

Theorem 2.6. *Let $\mathbf{G}$ be the collection of all admissible $(1,2)$-weighted cubic multigraphs under the partial order $\propto$ defined above. Then $P^* \propto (G, w)$ for every $(G, w) \in \mathbf{G}$ which has no cycle w-cover.*

Proof. It suffices to show that if (G, w) is an element of $\mathbf{G}$ which has no cycle w-cover and is minimal with respect to this property in the partial order $\propto$, then $P^* \propto (G, w)$.

Thus, any element of $\mathbf{G}$ that is properly smaller than (G, w) has a cycle w-cover. Let E^t be the set of edges of (G, w) of weight t, $t=1$ and 2. Since G is cubic and w is eulerian, the edges of E^1 form vertex-disjoint cycles of G and each vertex of G must be incident with at least one edge of weight 2.

We know that G is 2-edge-connected by the minimality assumption and admissibility. We claim that G is 3-edge-connected. To show this, assume that $\{e, e'\}$ is an edge-cut of G such that $G\backslash\{e, e'\}$ has two components A and B. Let $e=xy$ and $e'=x'y'$ with $x, x' \in V(A)$ and $y, y' \in V(B)$.

If $x=x'$, the other edge incident with x is a cut-edge of G, contradicting that G is 2-edge-connected. Similarly, $y \neq y'$ and we see that x, x', y and y' are four distinct vertices.

Since A and B are connected, $G_A = A \cup \{xx'\}$ and $G_B = B \cup \{yy'\}$ are 2-edge-connected cubic multigraphs. Assign weights to G_A and G_B by giving any edge f that belongs to G the weight $w(f)$ and assign $w(e)$ to both xx' and yy' (note that $w(e)=w(e')$). Call the resulting weighted multigraphs (G_A, w) and (G_B, w). Both are properly smaller than G and belong to $\mathbf{G}$. Hence, there are families $\mathbf{F}_A$ and $\mathbf{F}_B$ of cycles forming cycle w-covers of (G_A, w) and (G_B, w), respectively. Since xx' and yy' have the same weight, we can remove them from cycles of $\mathbf{F}_A$ and $\mathbf{F}_B$ in which they occur and connect the resulting paths, paired appropriately, with the edges xy and $x'y'$ to form cycles in G. These cycles together with the remaining cycles of $\mathbf{F}_A \cup \mathbf{F}_B$ form a cycle w-cover of G. This establishes the claim that G is 3-edge-connected.

We now claim that E^1 is a union of two odd length cycles D_1 and D_2, and any edge $xy \in E^2$ joins a vertex in $V(D_1)$ to a vertex in $V(D_2)$. Let $e=xy$ be an edge of weight 2. By the first claim, $G\backslash e$ is 2-edge-connected. The restriction of w to $G\backslash e$ is eulerian, implying that $\Pi(G\backslash e, w)$ is an admissible $(1,2)$-weighted cubic multigraph which is properly smaller than (G, w). It has a cycle w-cover, and, by Proposition 2.3, so does $(G\backslash e, w)$. Let $\mathbf{F}$ be a cycle w-cover of $(G\backslash e, w)$. Let $\mathbf{R}=\{C_0, C_1, \ldots, C_t\}$ be a minimal set of cycles from $\mathbf{F}$ such that their union is connected and contains both x and y (it is possible that $\mathbf{R}$ contains only C_0). By the minimality of $\mathbf{R}$, it is clear that we may assume that $x \in C_0$, $y \in C_t$ and $V(C_i) \cap V(C_j) = \emptyset$ for $C_i, C_j \in \mathbf{R}$, $i \neq j$ and $i \neq j \pm 1$.

We now define a new $(1,2)$-eulerian weighted multigraph (G', w') as follows. The union of the cycles $C_0, C_1, \ldots, C_t$ together with the edge xy form the multigraph G'; $w'(f), f \neq e$, is the number of cycles of $\mathbf{R}$ containing the edge f of G' and $w'(e)=2$. Clearly, $\Pi(G', w') \propto (G, w)$, $\Pi(G', w')$ is cubic and w' is an admissible $(1,2)$-weight since G' is 2-edge-connected.

If $\Pi(G', w')$ is properly smaller than (G, w), then $\Pi(G', w')$ has a cycle w'-cover and by Proposition 2.3, (G', w') also has a cycle w'-cover $\mathbf{F}'$. Then $\mathbf{F}' \cup (\mathbf{F} \backslash \mathbf{R})$ would be a cycle w-cover of G. Thus, we conclude that $(G', w') = (G, w)$.

If $t=0$, G must be the complete multigraph on two vertices with three parallel edges. It is easy to see that this multigraph has a cycle w-cover for any admissible $(1,2)$-weight. Thus, we assume $t>0$. Let

$$E_0 = \{f \in E(C_i)\colon i \text{ is even}\}$$

and

$$E_1 = \{f \in E(C_i): i \text{ is odd}\}.$$

Then color all edges of $E_0 \setminus E_1$ by yellow, all edges of $E_1 \setminus E_0$ by blue and all edges of $E_0 \cap E_1$ by green. This defines in an obvious way a proper 3-edge-coloring of $\Pi(G \setminus e)$ since it is cubic. However, it is not a proper 3-edge-coloring of $G \setminus e$ because both edges incident with x have the same color and both edges incident with y have the same color. No edge incident with x or y can be colored green because each of x and y lie in only one cycle of $\boldsymbol{R}$, respectively (by the minimality of $\boldsymbol{R}$). Obviously, the set of all yellow- and blue-colored edges is exactly the set of all weight 1 edges, and the green-colored edges are the edges of weight 2 of G (other than e).

If x and y lie in the same component of the subgraph induced by the blue and yellow edges of $G \setminus e$, then there exists a blue–yellow path P joining them. Exchange the roles of blue and yellow along P and color the edge e green. This gives us a proper 3-edge-coloring of G, which, by Lemma 2.1, would imply that (G, w) has a cycle w-cover, but this is a contradiction.

Therefore, x and y lie in different components of the subgraph induced by the blue and yellow edges of $G \setminus e$. Both of these two components are odd length cycles in G and all other components must be even length cycles. On the other hand, the weight-1 edges of G are exactly the edges of $G \setminus e$ that are colored blue and yellow and e is an arbitrary weight-2 edge of G. This establishes the claim that G is a σ-prism over an odd length cycle, with the edges of the two odd length cycles having weight 1 and the edges joining the two cycles having weight 2.

Since G is not 3-edge-colorable, it contains a subdivision of the Petersen graph by Theorem 2.5. If G is not the Petersen graph, then $P^* \propto (G, w)$ follows by definition. If G is the Petersen graph, it is easy to verify that the only admissible $(1,2)$-weight without a cycle w-cover gives P^*. □

3. Conclusions

The following result is an immediate corollary of Theorem 2.6.

Corollary 3.1. *Any 2-connected cubic multigraph that does not contain a subdivision of the Petersen graph has a cycle double cover.*

It is known that a minimum counterexample to the cycle double cover conjecture must be a cyclically 4-edge-connected cubic graph [8, 9]. This means that the following corollary is immediate.

Corollary 3.2. *A minimum counterexample to the cycle double-cover conjecture contains a subdivision of the Petersen graph.*

The following result was proved by Seymour [9] using the four-color theorem. Fleischner [5] has given a proof that does not use the four-color theorem and in a recent paper, Fleischner and Frank [6] have proved a generalization without employing the four-color theorem. Since it now follows easily from the results of Section 2, we also give a proof that does not use the four-color theorem.

Theorem 3.3. *If G is a 2-connected planar multigraph and w is an admissible weight on G, then (G, w) has a cycle w-cover.*

Proof. Seymour proves this result by establishing, without the use of the four-color theorem, that if there is a counterexample, then there is a cubic counterexample with an admissible $(1, 2)$-weight. At this point, he employs the four-color theorem to complete the proof. Instead, we observe that no such counterexample can exist by Theorem 2.6 because no planar multigraph can contain a subdivision of the Petersen graph. □

The minimum cycle cover problem for a multigraph is to find a family of cycles such that each edge of G lies in at least one cycle and the sum of the lengths of the cycles is minimum. The Chinese postman problem is discussed in many graph theory books (see [2]). Guan and Fleischner [4] showed that, for any 2-edge-connected planar graph, finding an optimal solution of the Chinese postman problem is equivalent to finding a solution of the minimum cycle cover problem. Their result may be obtained as a corollary of Theorem 3.3 by using edges of weights 1 and 2. Itai and Rodeh [7] pointed out that the two problems are not equivalent for arbitrary graphs. This suggests the following natural question. Are the Chinese postman problem and the minimum cycle covering problem equivalent for a graph that does not contain a subdivision of the Petersen graph? We can say something about the cubic case of this question. The same argument used to prove Guan and Fleischner's result mentioned above together with Theorem 2.6 also establishes the following result.

Theorem 3.4. *For any 2-connected cubic multigraph not containing a subdivision of the Petersen graph, the Chinese postman problem and the minimum cycle cover problem are equivalent.*

Recently, Theorem 2.6 has been generalized [1] to eliminate the restrictions on the multigraph being cubic and the weights being 1 and 2. The work was motivated by the result in the present paper and the proof uses essentially Theorem 2.6 of this paper as a starting point.

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