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Nowhere-zero 3-flows of highly connected graphs

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Abstract

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Let G be a k-edge-connected graph of order n. If $k \geqslant 4\lceil \log_2 n \rceil$ then G has a nowhere-zero 3-flow.

We use the notations of [2]. Let $G=(V, E)$ be a graph with vertex set V and edge set E. An *even subgraph* of G is a subgraph H of G such that the degree of each vertex is even in H. An *orientation* D of G is an assignment of a direction to each edge. A *weight function* f on $E(G)$ is an assignment of an integer $f(e)$ to each edge e. A *k-flow* of G is a pair (D, f), consisting of an orientation D and a weight function f, such that

(1) $-k<f(e)<k$, for each edge e;

(2) at every vertex v the net outflow of f is zero, that is the sum of f-values of edges with initial end v equals the sum of f-values of the edges with terminal end v.

(Refer to [12] and [6] for properties of integer flows.) The *support* of a k-flow is the set of all edges with nonzero weights. A *nowhere-zero k-flow* is a k-flow such that $f(e) \neq 0$ for every edge e of G.

Tutte's Conjecture (The 3-flow conjecture [9, 10, 5]). Every 2-edge-connected graph without 3-edge-cut has a nowhere-zero 3-flow.

Jaeger's Conjecture (The weak 3-flow conjecture [6]). There is an integer k such that every k-edge-connected graph has a nowhere-zero 3-flow.

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Previous results. (A) (Jaeger [5]). *A cubic graph has a nowhere-zero 3-flow if and only if it is bipartite.*

(B) (Jaeger [5]). *Every 4-edge-connected graph has a nowhere-zero 4-flow.*

(C) (Grötzsch [4] or see [6, p. 79] and [10]). *Every 2-edge-connected planar graph without 3-edge-cut has a nowhere-zero 3-flow.*

(D) (Grünbaum [3] and Aksionov [1]). *Every 2-edge-connected planar graph with at most three 3-cuts has a nowhere-zero 3-flow.*

(E) (Steinberg and Younger [10]). *Every 2-edge-connected graph with at most one 3-cut that can be embedded in the projective plane has a nowhere-zero 3-flow.*

The following theorem is the main result of this paper.

Theorem. *Let G be a k-edge-connected graph with t odd vertices. If $k \geq 4\lceil \log_2 t \rceil$, then G has a nowhere-zero 3-flow.*

Corollary. *Let G be a k-edge-connected graph of order n. If $k \geq 4\lceil \log_2 n \rceil$, then G has a nowhere-zero 3-flow.*

The following lemmas will be used in the proof of the main theorem.

Lemma 1 (Nash-Williams [8] and Tutte [11], or see [7] or [2, p. 31]). *Every $2k$-edge-connected graph contains k edge-disjoint spanning trees.*

The set of odd-degree vertices of a graph G is denoted by $O(G)$. A subgraph H of G is called a *parity subgraph* of G if $O(H) = O(G)$. A proof of the following well-known lemma will be given for the sake of completeness.

Lemma 2. *Every spanning tree of a connected graph G contains a parity subgraph of G.*

Proof. Let T be a spanning tree of G. For every edge e in $E(G)\setminus E(T)$, let C_e be the unique cycle contained in $T \cup \{e\}$. The symmetric difference (binary sum) of C_e's for all e in $E(G)\setminus E(T)$ is an even subgraph H of G and H contains all edges of $E(G)\setminus E(T)$. Thus $G\setminus E(H)$ is a parity subgraph of G contained in T. □

Let H be a graph with a 3-flow (D, f). The support of f is denoted by $H_{f\neq 0}$ or $\mathrm{Sup}(f)$ if no confusion occurs and the subgraph of H induced by all edges with value zero in f are denoted by $H_{f=0}$. The following lemma plays a central role in the proof of the main theorem.

Lemma 3. *Let T_1, T_2 and T_3 be three edge-disjoint parity subgraphs of G and let H be the subgraph of G induced by the edge set $E(T_1 \cup T_2 \cup T_3)$. Then H has a 3-flow (D, f) such that $|O(H_{f=0})| \leq \frac{1}{2}|O(G)|$.*

Proof. Let R_3 be a minimal parity subgraph of G contained in T_3. It is obvious that $T_3 \backslash E(R_3)$ is an even subgraph of H and hence is the support of a 2-flow. So it is sufficient to show that $H' = E(T_1 \cup T_2 \cup R_3) = H \backslash [E(T_3) \backslash E(R_3)]$ has a 3-flow satisfying the lemma. Since it is minimal, the parity subgraph R_3 is acyclic and therefore is a union of edge-disjoint paths $P_1, \ldots, P_t$ such that each P_μ joins a pair of odd vertices $v_{2\mu-1}$ and $v_{2\mu}$ of G where $O(G) = \{v_1, \ldots, v_{2t}\}$. Construct an even graph S_i for $i = 1, 2$ by adding edges $v_{2\mu-1}v_{2\mu}$ to T_i for each $\mu = 1, \ldots, t$.

Assign an orientation to $E(T_1)$, $E(T_2)$ and paths $P_1, \ldots, P_t$. And let the direction of ech edge in P_μ and the direction of the new edges $v_{2\mu-1}v_{2\mu}$ in each S_i be the same as that of the path P_μ for each $\mu = 1, \ldots, t$. Let D denote the resulting orientation.

Since each S_i is even, let (D, f_i) be a nowhere-zero 2-flow of S_i. Let S_i^* be the even subgraph of G obtained by replacing each edge $v_{2\mu-1}v_{2\mu}$ by the path P_μ for $\mu = 1, \ldots, t$. The flow (D, f_i) defines in the obvious way a nowhere-zero 2-flow of S_i^* for $i = 1, 2$ which we also denote by (D, f_i). Then $(D, f_1 + f_2)$ is a 3-flow of H'. It is obvious that $H'_{f_1+f_2=0}$ is the union of some paths $P_{i_1}, \ldots, P_{i_r}$. If $r \leq t/2$, then

$$\left|O\left(\bigcup_{\mu=1}^{r} P_{i_\mu}\right)\right| = 2r \leq t = \frac{|O(G)|}{2}.$$

Otherwise, considering the 3-flow $(D, f_1 - f_2)$, we see that $H'_{f_1-f_2=0}$ is the union of the paths in $\{P_1, \ldots, P_t\} \backslash \{P_{i_1}, \ldots, P_{i_r}\}$ and has $2t - 2r$ $(2t - 2r < t = \frac{1}{2}|O(G)|)$ odd vertices. □

Lemma 4. *Let $T_0, \ldots, T_{2s-1}$ be edge-disjoint subgraphs of a connected graph G where T_0 is a parity subgraph of G and $T_1, \ldots, T_{2s-1}$ are spanning trees of G. If $|O(G)| \leq 2^s$, then G has a nowhere zero 3-flow.*

Proof. The following basic property of graphs will be used to verify the cases of $s = 0$ and $s = 1$,

The number of odd vertices in any graph is even. (*)

When $s = 0$ the graph G is an even graph by (*), and hence the graph G admits a nowhere-zero 2-flow. When $s = 1$, assume that $O(G) = \{x, y\}$. By (*), x and y are contained in the same component of T_0 and T_1 and therefore any edge-cut separating x and y must be of order at least two. By (*) again, any edge-cut separating x and y must be of odd order. Thus, by Menger's Theorem, there are three edge-disjoint (x, y)-paths P_1, P_2 and P_3 in G. Let $P_\mu = v_1^\mu \cdots v_{r_\mu}^\mu$ where $v_1^\mu = x$ and $v_{r_\mu}^\mu = y$ for $\mu = 1, 2, 3$. Assign a flow (D_1, f_1) on the induced subgraph

$G(E(P_1 \cup P_2 \cup P_3))$ such that

$$v_i^\mu \to v_{i+1}^\mu$$

for each edge of $G(E(P_1 \cup P_2 \cup P_3))$ and

$$f_1(e) = \begin{cases} 1 & \text{if } e \in P_1 \cup P_2, \\ -2 & \text{if } e \in P_3. \end{cases}$$

So (D_1, f_1) is a nowhere-zero 3-flow of $G(E(P_1 \cup P_2 \cup P_3))$. Since $G \backslash E(P_1 \cup P_2 \cup P_3)$ is even, it has a nowhere-zero 2-flow (D_2, f_2) and hence the graph G has a nowhere-zero 3-flow $(D_1 + D_2, f_1 + f_2)$.

Let $s \geq 2$. We proceed by induction on s. Let R_i be a parity subgraph contained in T_i for $i = 0, 1, 2$. By Lemma 3, let f_1 be a 3-flow of $H = G(E(R_0 \cup R_1 \cup R_2))$ such that $|O(H_{f=0})| \leq |O(H)/2|$. Let $G' = G \backslash E(H_{f \neq 0})$. Since $G' = [G \backslash E(H)] \cup E(H_{f=0})$ and $G \backslash E(H)$ is an even subgraph of G, $H_{f=0}$ is a parity subgraph of G'. Note that $|O(G')| \leq |O(G)/2| \leq 2^{s-1}$ and $H_{f=0}, T_3, \ldots, T_{2s-1}$ are edge-disjoint subgraphs of G'. By inductive hypothesis, G' has a nowhere-zero 3-flow f'. Thus $f + f'$ is a nowhere-zero 3-flow of G since $\mathrm{Sup}(f) \cap \mathrm{Sup}(f') = \emptyset$. □

Proof of the Theorem. Let $2^{s-1} < t \leq 2^s$ (that is, $s = \lceil \log_2 t \rceil$). By Lemma 1, the graph G contains at least $2s$ edge-disjoint spanning trees. Then the main theorem is an immediate corollary of Lemma 4. □

The main theorem in this paper established a relation between the edge-connectivity and a number of odd vertices of a graph which guarantees the existence of a nowhere-zero 3-flow. the method applied in the proof of Lemma 4 could be used to prove the weak 3-flow conjecture if the following conjecture could be verified.

Conjecture. There is a pair of 'large' integers a and b such that any graph G, with $|O(G)| \leq |V(G)|/a$ and containing b edge-disjoint spanning trees, must have a nowhere-zero 3-flow.

Let $a \leq 2^c$. Let G be a $2k$-edge-connected graph where $k \geq b + 2c$. By Lemma 1, G contains at least k edge-disjoint spanning trees $T_0, \ldots, T_{k-1}$. Repeating the inductive argument in the proof of Lemma 4, we obtain a parity subgraph H such that

$$E(H) \subseteq \bigcup_{i=0}^{2c-1} E(T_i)$$

and a 3-flow f with support in H and

$$|O(H_{f=0})| \leq \frac{|O(G)|}{2^c}.$$

Consider the spanning subgraph $G' = G \setminus E(H_{f \neq 0})$ which has at least b edge-disjoint spanning trees and has at most $|V(G')|/2^c$ odd vertices. If the above conjecture were verified, then G' would have a nowhere-zero 3-flow f' and therefore G would have a nowhere-zero 3-flow $f' + f$.

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