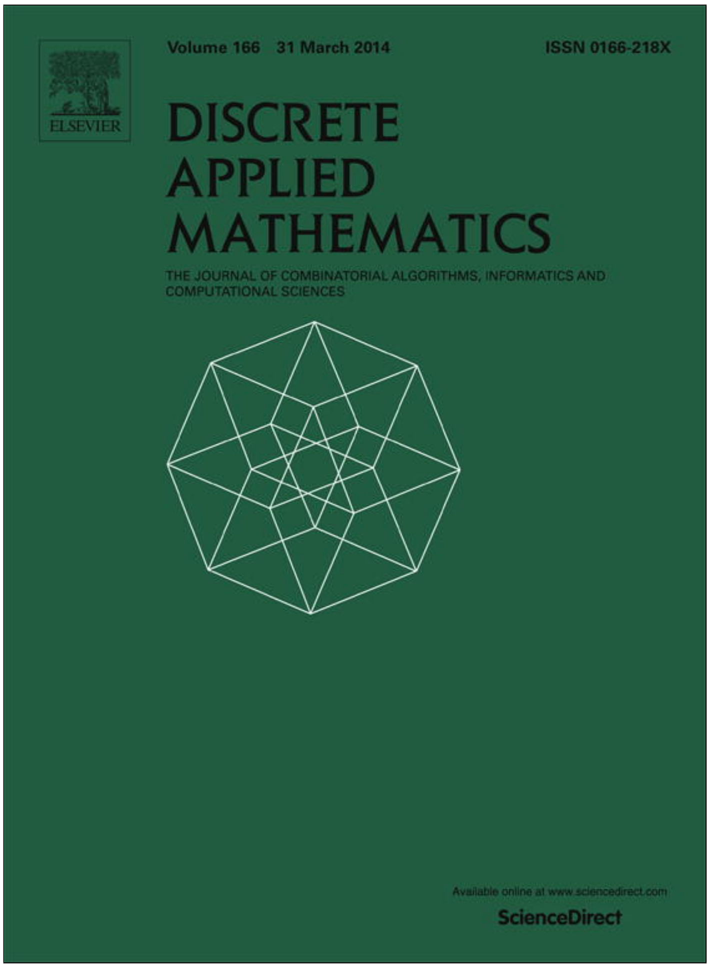
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Note

Perfect matching covering, the Berge–Fulkerson conjecture, and the Fan–Raspaud conjecture[☆]

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A B S T R A C T

Let m_t^* be the largest rational number such that every bridgeless cubic graph G associated with a positive weight ω has t perfect matchings $\{M_1, \ldots, M_t\}$ with $\omega(\cup_{i=1}^t M_i) \geq m_t^* \, \omega(G)$. It is conjectured in this paper that $m_3^* = \frac{4}{5}$, $m_4^* = \frac{14}{15}$, and $m_5^* = 1$, which are called the weighted PM-covering conjectures. The counterparts of this new invariant m_t^* and conjectures for unweighted cubic graphs were introduced by Kaiser et al. (2006). It is observed in this paper that the Berge–Fulkerson conjecture implies the weighted PM-covering conjectures. Each of the weighted PM-covering conjectures is further proved to imply the Fan–Raspaud conjecture. Furthermore, a 3PM-coverage index τ (respectively, τ^* for the weighted case) is introduced for measuring the maximum ratio of the number of (respectively, the total weight of) edges covered by three perfect matchings in bridgeless cubic graphs and assessing how far a snark is from being 3-edge-colorable. It is proved that the determination of τ^* for bridgeless cubic graphs is an *NP*-complete problem.

1. Introduction

The following are two of the major open problems in graph theory.

Conjecture 1.1 (*Berge and Fulkerson [2]*). *Every bridgeless cubic graph G has six perfect matchings such that each edge of G is covered by exactly two of them.*

Conjecture 1.2 (*Fan and Raspaud [1]*). *Every bridgeless cubic graph G contains three perfect matchings M_1, M_2, M_3, such that no edge is covered by all of them.*

Following the definitions introduced in [4], we define $m_t(G)$ and m_t as follows.

$$m_t(G) = \max\left\{ \frac{|\cup_{i=1}^t M_i|}{|E(G)|} \;\middle|\; \text{for all sets } \{M_1, \ldots, M_t\} \text{ of } t \text{ perfect matchings of } G \right\};$$
$$m_t = \inf\{m_t(G) \mid \text{for all bridgeless cubic graphs } G\}.$$

Let P_{10} denote the Petersen graph. Kaiser et al. proposed the following conjecture.

Conjecture 1.3 (*Kaiser, Král and Norine [4]*). *For $i = 3, 4, 5$, $m_i = m_i(P_{10})$.*

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The following concepts and conjectures generalize the definitions of $m_t(G)$, m_t, and Conjecture 1.3 for weighted cubic graphs.

Definition 1.1. Let t be a positive integer, and let G be a bridgeless cubic graph associated with a weight $\omega : E(G) \mapsto R^+$, where R^+ denotes the set of positive real numbers. Define

$$m_t^*(G;\omega) = \max\left\{ \frac{\sum_{e\in\cup_{i=1}^{t} M_i} \omega(e)}{\sum_{e\in E(G)} \omega(e)} \,\middle|\, \text{for all sets } \{M_1,\ldots,M_t\} \text{ of } t \text{ perfect matchings of } G \right\};$$

$$m_t^*(G) = \inf\{m_t^*(G;\omega) \mid \text{for all weights } \omega : E(G) \mapsto R^+\}$$

and

$$m_t^* = \inf\{m_t^*(G) \mid \text{for all bridgeless cubic graphs } G\}.$$

Similar to [7] for the unweighted case, we observe that $m_3^*(P_{10}) = \frac{4}{5}$, $m_4^*(P_{10}) = \frac{14}{15}$, and $m_5^*(P_{10}) = 1$. So we propose the following conjectures.

Conjecture 1.4 (*Weighted PM-Covering Conjectures*)**.** (a) $m_3^* = \frac{4}{5}$; (b) $m_4^* = \frac{14}{15}$; (c) $m_5^* = 1$.

Obviously, Conjecture 1.4 implies Conjecture 1.3. Kaiser et al. [4] have proved that $m_2 = \frac{3}{5}$, $\frac{27}{35} \le m_3 \le \frac{4}{5}$. These results have been extended by Mazzuoccolo in [5] to $\frac{55}{63} \le m_4$; $\frac{215}{231} \le m_5$. Mazzuoccolo has also proved that the Berge–Fulkerson conjecture is equivalent to the Berge conjecture (the conjecture that $m_5 = 1$) [6]. It is proved by Patel in [7] that the Berge–Fulkerson conjecture implies Conjecture 1.3. Some of these early results are further extended for weighted graphs in this paper.

Proposition 1.5. *The Berge–Fulkerson conjecture implies Conjecture 1.4.*

Proposition 1.6. $m_1^* = m_1^*(P_{10}) = \frac{1}{3}$; $m_2^* = m_2^*(P_{10}) = \frac{3}{5}$; $\frac{27}{35} \le m_3^* \le \frac{4}{5}$; $\frac{55}{63} \le m_4^*$; $\frac{215}{231} \le m_5^*$.

The relation between the weighted PM-covering conjecture and the Fan–Raspaud conjecture is studied.

Theorem 1.7. *If Conjecture 1.4-(a) is true, then the Fan–Raspaud conjecture is also true.*

Theorem 1.8. *If Conjecture 1.4-(b) is true, then the Fan–Raspaud conjecture is also true.*

In this paper, we define the 3PM-coverage index (respectively, weighted 3PM-coverage index) of a graph G as $\tau(G) = m_3(G)$ (respectively, $\tau^*(G) = m_3^*(G)$).

Theorem 1.9. *The determination of $\tau^*(G)$ is an NP-complete problem.*

Proposition 1.5 is a straightforward observation. Let $\mathcal{M}$ be the set of six perfect matching double cover of G. Each m_k^* $(1 \le k \le 6)$ is calculated by taking the average of the coverages of all k-subsets of $\mathcal{M}$ (see [7] for the unweighted case).

Proposition 1.6 can be proved by a similar argument as in the proof of Theorem 3.1 of [5]. In order to shorten the paper, the proof is omitted.

The remainder of this paper is organized as follows. In Section 2, Theorems 2.4 and 2.5 are proved. In Section 3, Theorem 3.3 is proved.

2. Perfect matching covering for weighted graphs

The following two lemmas are well known and will be used in the proofs in this paper.

Lemma 2.1. *Let G be a bridgeless cubic graph, and let T be an edge-cut. For any perfect matching M of G, we have $|M \cap T| \equiv |T| (\text{mod } 2)$.*

Lemma 2.2. *In the Petersen graph P_{10}, we have the following properties.*

(1) *P_{10} has precisely six different perfect matchings.*
(2) *The intersection of each pair of distinct perfect matchings has exactly one edge.*
(3) *The intersection of any three distinct perfect matchings is empty.*
(4) *Let k be an integer between 2 and 5. Then the union of any k distinct perfect matchings has $15 - \frac{(6-k)(5-k)}{2}$ edges.*

Parts (1)–(3) of Lemma 2.2 can be found in [8] (also see [9]). Part (4) is a corollary of (1)–(3).

2.1. The Fan–Raspaud conjecture and PM-covering conjectures

2.1.1. $\oplus_2$-sum with the Petersen graph

Let G_1, G_2 be two bridgeless cubic graphs, and let $e_1 = uv \in E(G_1)$, $e_2 = xy \in E(G_2)$. Construct a new graph, called the $\oplus_2$-sum of G_1 and G_2 associated with e_1 and e_2.

$$G_1 \oplus_2 G_2 = [G_1 - \{e_1\}] \cup [G_2 - \{e_2\}] \cup \{ux, vy\}.$$

The two new edges ux and vy are called *the substitution edges* of uv or xy in $G_1 \oplus_2 G_2$, and e_1 and e_2 are called the *pairing edges* of each other. Note that P_{10} is edge-transitive; $G \oplus_2 P_{10}^e$ is used to denote the $\oplus_2$-sum of G and P_{10} associated with e_1 and e, where $e \in E(G)$, and P_{10}^e is also used to denote a copy of P_{10} in the above operation.

Let G be a cubic bridgeless graph with m edges. Denote by $\{e_1, e_2, \ldots, e_m\}$ the edge set of G. Let

$$G_1 = G \oplus_2 P_{10}^{e_1}, G_2 = G_1 \oplus_2 P_{10}^{e_2}, \ldots, G_m = G_{m-1} \oplus_2 P_{10}^{e_m}.$$

Then G_m is called the $\oplus_2$-*sum full composition graph of G with P_{10}*, denoted by $G \oplus_2^F P_{10}$.

Lemma 2.3. *Given a cubic bridgeless graph G, let $G' = G \oplus_2^F P_{10}$, and let $\{M_1', \ldots, M_t'\}$ be a set of t perfect matchings of G'. We have the following observations.*

(1) *For each $e \in E(G)$, any perfect matching M_i' of G' must either contain both substitution edges of e or contain none of them.*

(2) *For each $e \in E(G)$, by identifying all vertices of $V(G') - V(P_{10}^e)$ and suppressing the resulting degree-2 vertices, a perfect matching M_i^e of P_{10}^e can be obtained from the perfect matching M_i' of G'.*

(3) *By contracting all non-substitution-edges of G' and suppressing all resulting degree-2 vertices, a perfect matching M_i of G can be obtained from the perfect matching M_i' of G'.*

Proof. Part (1) is an immediate corollary of Lemma 2.1. Parts (2) and (3) are implied by (1). □

2.1.2. Weighted PM-covering conjectures and the Fan–Raspaud conjecture

Theorem 2.4. *If Conjecture 1.4-(a) is true, then the Fan–Raspaud conjecture is also true.*

Proof. Given any bridgeless cubic graph G, define a weight function ω' on $G' = G \oplus_2^F P_{10}$ as follows:

$$\omega'(f) = \begin{cases} \dfrac{1}{2} & \text{if } f \text{ is a substitution edge of an edge } e \in E(G) \\ 1 & \text{if } f \text{ is not a substitution edge.} \end{cases}$$

By the assumption that Conjecture 1.4-(a) is true, for the newly defined weight function, there are three perfect matchings M_1', M_2', M_3' in G' such that $\omega'(\cup_{i=1}^3 M_i') \geq \frac{4}{5}\, \omega'(G') = 12|E(G)|$.

For each edge e of G and any perfect matching M in G', by Lemma 2.1 the two substitution edges of e in G' are both or neither covered by M. Let E_0 be the set of edges in G whose two substitution edges in G' are both covered by all three perfect matchings $\{M_1', M_2', M_3'\}$.

Let $\{M_1, M_2, M_3\}$ be the set of perfect matchings of G corresponding to $\{M_1', M_2', M_3'\}$ (see Lemma 2.3(3)). Here, $E_0 = \bigcap_{i=1}^3 M_i$, and we are to show that $E_0 = \emptyset$.

For each $e \in E_0 = \bigcap_{i=1}^3 M_i$, let $\{M_1^e, M_2^e, M_3^e\}$ be the set of perfect matchings of P_{10}^e corresponding to $\{M_1', M_2', M_3'\}$ (see Lemma 2.3-(2)). Therefore, those three perfect matchings in P_{10}^e intersect in at least one edge, and $|\{M_1^e, M_2^e, M_3^e\}| \leq 2$, since there are no three distinct perfect matchings in the Petersen graph with nonempty intersection by Lemma 2.2-(2).

By Lemma 2.2-(4), two distinct perfect matchings cover exactly 9 edges of the Petersen graph and three distinct perfect matchings cover exactly 12 edges. We have that

$$\omega'(\cup_{i=1}^3 M_i') \leq 9|E_0| + 12(|E(G)| - |E_0|) = 12|E(G)| - 3|E_0|.$$

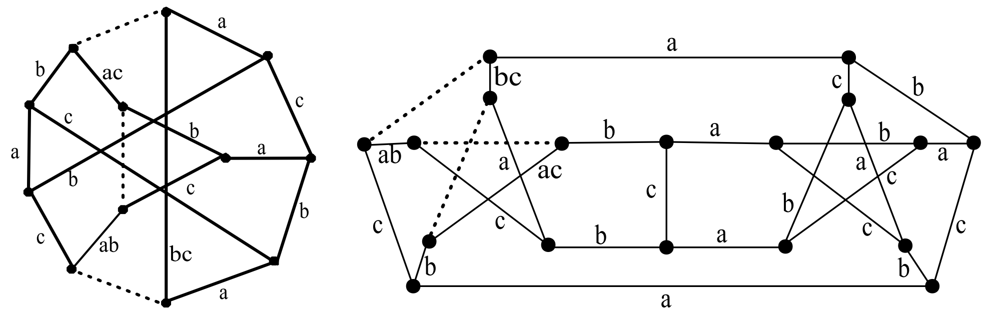
Then, by the fact that $\omega'(\cup_{i=1}^3 M_i') \geq 12|E(G)|$, we have $|E_0| = 0$, as we desired. □

Theorem 2.5. *If Conjecture 1.4-(b) is true, then the Fan–Raspaud conjecture is also true.*

Proof. The proof is similar to the argument used in above theorem. The only difference is the definition of edge set E_0, which consists of all edges in G whose two substitution edges are covered by three or four perfect matchings in G'. We shall get a similar inequality

$$14\,|E(G)| \leq \omega'(\cup_{i=1}^4 M_i') \leq 12|E_0| + 14(|E(G)| - |E_0|),$$

which implies that $|E_0| = 0$, as we desired. □



(a) Tietze's snark. (b) Blanuša's snark.

Fig. 1. Two snarks with 3PM-coverage indices $\frac{5}{6}$ and $\frac{8}{9}$, respectively.

3. 3PM-coverage index—a rank for snarks

Definition 3.1. Let G be a bridgeless cubic graph. We call $m_3(G)$ the 3*PM-coverage index* of G, denoted by $\tau(G)$; similarly, we call $m_3^*(G)$ the weighted 3*PM-coverage index* of G, denoted by $\tau^*(G)$.

According to Conjectures 1.3 and 1.4-(a), the lower bound for both τ and τ^* is expected to be at least $\frac{4}{5}$. A graph has the 3*PM*-coverage index $\tau = 1$ if and only if it is 3-edge-colorable. Hence, the graph invariant τ (as well as its weighted version τ^*) can be considered as a ranking for snarks.

Fig. 1 illustrates two examples with 3PM-coverage indices between $\frac{4}{5}$ and 1. For the Tietze snark T, $\tau(T) = \frac{5}{6}$ and, for the Blanuša snark B, $\tau(B) = \frac{8}{9}$.

In Fig. 1, each snark has a set of three perfect matchings $\{M_a, M_b, M_c\}$, where edges labeled with x ($x \in \{a, b, c\}$) belong to the matching M_x, and dashed edges are not covered by $M_a \cup M_b \cup M_c$.

Just like the determination of 3-edge-colorability [3], the determination of the invariant τ^* is also an *NP*-complete problem (see Theorem 3.3).

Instance 3.1 $((G, \omega, \eta))$**.** *A bridgeless cubic graph G associated with a positive weight $\omega : E(G) \mapsto R^+$, and a real number η: $\frac{4}{5} < \eta < 1$.*

Problem 3.2. Does the graph G have a set of three perfect matchings $\{M_1, M_2, M_3\}$ with the property that $\omega(\cup_{i=1}^3 M_i) \geq \eta\,\omega(G)$?

Theorem 3.3. *Problem 3.2 is an NP-complete problem.*

Proof. Suppose that there exists a polynomial-time algorithm $\mathcal{P}$ for Problem 3.2.

Taks as given a cubic bridgeless graph G with m edges and a real number η with $\frac{4}{5} < \eta < 1$. Let $\epsilon = \frac{15\eta-12}{1.2}$. Let $G' = G \oplus_2^F P_{10}$. Define the weight function ω' on G' as follows.

$$\omega'(f) = \begin{cases} \dfrac{1}{2} & \text{if } f \text{ is a substitution edge of some } e \in E(G); \\ 1+\epsilon & \text{if } f \text{ is incident with a substitution edges of } e; \\ 1-0.4\,\epsilon & \text{else.} \end{cases}$$

Since $\frac{4}{5} < \eta < 1$, $0 < \epsilon < 2.5$, ω' is a positive weight of G'. By the construction of G' and ω', $\omega'(G') = 15m$.

Since G' is also a bridgeless cubic graph, $\mathcal{P}$ can determine if there exist three perfect matchings $\{M_1', M_2', M_3'\}$ covering edges of G' such that $\omega'(\cup_{i=1}^3 M_i') \geq \eta\omega'(G') = \frac{12+1.2\epsilon}{15}\omega'(G') = (12 + 1.2\,\epsilon)\,m$.

Case 1. $\mathcal{P}$ determines that there exist three perfect matchings $\{M_1', M_2', M_3'\}$ of G' such that $\omega'(\cup_{i=1}^3 M_i') \geq \eta\omega'(G')$ $(= \frac{12+1.2\epsilon}{15}\omega'(G') = (12 + 1.2\,\epsilon)\,m)$.

By Lemma 2.3(3), let $\{M_1^e, M_2^e, M_3^e\}$ be the corresponding set of perfect matchings in P_{10}^e for each edge $e \in E(G)$. Suppose that e' is the pairing edge of e in P_{10}^e, and let $\omega'(e') = 1$.

Claim.

$$\sum_{f \in P_{10}^e \cap (\cup_{i=1}^3 M_i')} \omega'(f) = \sum_{f \in \cup_{i=1}^3 M_i^e} \omega'(f) \leq (12 + 1.2\epsilon).$$

The equality holds if and only if each substitution edge of e and all edges in P_{10}^e incident with substitution edges of e are covered precisely once by $\{M_1', M_2', M_3'\}$.

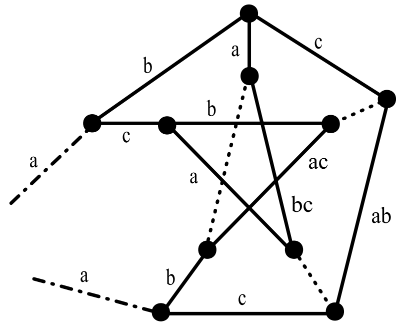


Fig. 2. Three perfect matchings $\{M_a, M_b, M_c\}$ reach the maximum coverage in P_{10}^e.

Proof of the Claim. By Lemma 2.2-(4), the set of perfect matchings $\{M_1^e, M_2^e, M_3^e\}$ covers at most 12 edges of P_{10}^e. So the total weight of the edges they cover is at most

$$15 - 3(1 - 0.4\epsilon) = 12 + 1.2\epsilon.$$

Since a least weighted edge is of weight $(1 - 0.4\epsilon)$ and those least weighted edges are not incident with substitution edges of e, the total weight $\omega'(\bigcup_{i=1}^3 M_i^e)$ reaches its maximum if and only if $\{M_1^e, M_2^e, M_3^e\}$ are distinct in P_{10}^e and miss precisely three edges of weight $(1 - 0.4\epsilon)$. (Fig. 2 is an example showing a set of three perfect matchings $\{M_a, M_b, M_c\}$ of P_{10}^e missing three dashed edges with weight $(1 - 0.4\epsilon)$.) This completes the proof of the claim.

Since $\omega'(\cup_{i=1}^3 M_i') \geq \frac{12+1.2\epsilon}{15}\omega'(G') = (12 + 1.2\,\epsilon)\,m$, by the above claim, each corresponding $\{M_1^e, M_2^e, M_3^e\}$ must reach its maximum total weight in P_{10}^e. Thus, we have the following.

(1) Each substitution edge of e is covered once in G'.
(2) All edges in P_{10}^e incident to substitution edges are also covered once.

By Lemma 2.3(3), the corresponding set of perfect matchings $\{M_1, M_2, M_3\}$ of G is a 1-factorization of G. This way we can find a 3-edge coloring of the graph G.

Case 2. If $\mathscr{P}$ determines that no set of three perfect matchings of G' has the property, then we claim that G is not 3-edge colorable.

Prove by contradiction. Let $\{M_1, M_2, M_3\}$ be a 1-factorization of G. Then we can construct three distinct perfect matchings $\{M_1', M_2', M_3'\}$ of G' just as in the above analysis. The total weight of edges covered by the three perfect matchings can be easily checked to be of ratio η of the total weight of G', a contradiction. $\square$

Note that the computational complexity of the unweighted invariant τ remains unknown.

Conjecture 3.4. *Given a real number $1 > \eta > \frac{4}{5}$, the decision problem whether $\tau(G) \geq \eta$ for an unweighted cubic bridgeless graph G is also an NP-complete problem.*

References

[1] G. Fan, A. Raspaud, Fulkerson's conjecture and circuit covers, J. Combin. Theory Ser. B 61 (1994) 133–138.
[2] D.R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, Math. Program. 1 (1971) 168–194.
[3] I. Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (1981) 718–720.
[4] T. Kaiser, D. Král, S. Norine, Unions of Perfect Matching in Cubic Graphs, Topics in Discrete Mathematics, in: Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 225–230.
[5] G. Mazzuoccolo, Covering a cubic graph with perfect matchings, Nov 2011. arXiv:1111.1871v2.
[6] G. Mazzuoccolo, The equivalence of two conjectures of Berge and Fulkerson, J. Graph Theory 68 (2011) 125–128.
[7] V. Patel, Unions of perfect matchings in cubic graphs and implications of the Berge–Fulkerson, CDAM Research Report, LSE-CDAM-2006-06.
[8] P.D. Seymour, Sums of circuits, in: J.A. Bondy, U.S.R. Murty (Eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, pp. 342–355.
[9] C.-Q. Zhang, Circuit Double Cover of Graphs, Cambridge Press, 2012.