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NOTE

ON CIRCULAR FLOWS OF GRAPHS

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A sufficient condition for graphs with circular flow index less than 4 is found in this paper. In particular, we give a simple proof of a result obtained by Galluccio and Goddyn *(Combinatorica, 2002)*, and obtain a larger family of such graphs.

We refer readers to [1], [2] and [7] for the standard terminology and notations in this paper.

The following theorem was proved by Galluccio and Goddyn.

Theorem 1 (Galluccio and Goddyn [2]). *Let G be a 6-edge-connected graph. Then the circular flow index of G, $\phi_C(G) < 4$.*

Here, we give a simple proof of this theorem without using linear programming.

Proof. Since G is 6-edge-connected, by Tutte [5, Theorem 1] or Nash-Williams [4, Theorem 1], let T_1, T_2, T_3 be three edge disjoint spanning trees of G. Let P_i be a parity subgraph of T_i (for $i = 1, 2$ only). Now fixing some orientation of G. Since $P_1 \cup P_2$ and $G \setminus E(P_1)$ are even graphs, let f_1 be a nowhere-zero 2-flow with support $E(P_1) \cup E(P_2)$ and f_2 be a nowhere-zero 2-flow with support $E(G) \setminus E(P_1)$. Then $f = f_1 + 2f_2$ is a nowhere-zero 4-flow of G. Reorient the edges of G such that the resulting correspondent 4-flow $f^* > 0$. We will show this is the required orientation. First, this orientation is

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a strong orientation because it is easy to show that each edge is contained in a directed circuits. For each nonempty proper subset $X \subset V(G)$, the edge cut $\delta(X)$ contains at least one edge in T_3, hence having flow value 2. Therefore

$$3|\delta^+(X)| \geq \text{outflow of } X = \text{inflow of } X \geq |\delta^-(X)|,$$

with at least one strict inequality, as there is an edge in the cut of flow value 2. By the definition of the circular flow index (see [2]), $\phi_C(G) < 4$. ∎

Similarly, we get the following results.

Theorem 2. *Let G be a graph. If G has a nontrivial parity subgraph decomposition such that at least one of its members is connected and spanning, then $\phi_c(G) < 4$.*

Theorem 3. *If a graph contains two edge-disjoint subgraphs P and H such that P is a parity subgraph and H is a connected, spanning collapsible subgraph of G, then $\phi_C(G) < 4$.*

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