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NOWHERE-ZERO 4-FLOWS, SIMULTANEOUS EDGE-COLORINGS, AND CRITICAL PARTIAL LATIN SQUARES

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It is proved in this paper that every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow. This result implies a conjecture originally proposed by Keedwell (1993) and reproposed by Cameron (1999) about simultaneous edge-colorings and critical partial Latin squares.

1. Introduction

The following result is the main theorem of the paper.

Theorem 1.1. *Every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow.*

A corollary of Theorem 1.1 solves a conjecture originally proposed by Keedwell [5] and reproposed by Cameron [2].

Theorem 1.2. (Keedwell-Cameron Conjecture) *Every bipartite graphic sequence S with the minimum degree $\delta(S) \geq 2$ has a realization G of S such that G has two proper edge-colorings with the following properties:*

(1) for any vertex, the set of colors appearing on edges at that vertex are the same in both colorings;

(2) no edge receives the same color in both colorings.

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The original conjecture was proposed by Keedwell [5] concerning the existence of critical partial Latin squares. A graph theory version of the conjecture (as described in Theorem 1.2) was reproposed by Cameron [2].

Theorem 1.1 was proved by Hajiaghaee *et al.* [7] for $\delta \geq 4$ and by Keedwell [5], Mahdian *et al.* [9] for some other special cases.

2. Notation and terminology

For technical reasons, multiple edges (parallel edges) are allowed in some cases in this paper, though the main result is for simple graphs only. A graph that may have multiple edges is called *a multigraph*.

A *circuit* is a 2-regular, connected subgraph and a *cycle* is the union of several edge-disjoint circuits.

Let $U_1, U_2 \subseteq V(G)$ with $U_1 \cap U_2 = \emptyset$. The set of all edges between U_1 and U_2 is denoted by $[U_1, U_2]$.

A path $v_0 v_1 \dots v_r$ of a graph G is called a *subdivided edge* if $d_G(v_i) = 2$ for each $i = 1, \dots, r-1$.

2.1. Graphic degree sequences

Let $S = \{s_1, \dots, s_m, t_1, \dots, t_n\}$ be a positive integer sequence with a partition $\{\{s_1, \dots, s_m\}, \{t_1, \dots, t_n\}\}$. The sequence S is called a *bipartite graphic sequence* if there is a bipartite graph G with bipartition $\{X, Y\}$ such that

$$\{d(x_1), \dots, d(x_m)\} = \{s_1, \dots, s_m\},$$

and

$$\{d(y_1), \dots, d(y_n)\} = \{t_1, \dots, t_n\}$$

where $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ and $d(v)$ is the degree of a vertex v; the graph G is called a *realization of* S.

A sequence is a bipartite graphic sequence if and only if it satisfies the Gale-Ryser condition [4], [10] (or see [13] Theorem 4.3.14). Since the Gale-Ryser condition is not to be applied in the proof of the main result, we omit the detail here.

2.2. Bipartite multigraphs

Definition 2.1. Let G be a bipartite multigraph with the bipartition $\{X, Y\}$.

(I) Let $x \in X$ and $y \in Y$. The *multiplicity of G between the vertices x and y*, denoted by $m_G(x,y)$, is the number of edges of G between the vertices x and y.

(II) Let

$$\mu : X \times Y \mapsto Z^+$$

be a function. We say that *the multigraph G is upper bounded by μ* if

$$m_G(x,y) \le \mu(x,y)$$

for every $x \in X$ and every $y \in Y$.

Definition 2.2. Let G be a bipartite multigraph with the bipartition $\{X,Y\}$ and upper bounded by a function μ.

(I) One designated edge $e_0 = x_0y_0 \in E(G)$ is called *the special edge of the ordered triple* (G,μ,e_0) and satisfies

$$\mu(x,y) = 1$$

for every $x \in X \setminus \{x_0\}$ and every $y \in Y \setminus \{y_0\}$.

(II) A bipartite multigraph H with the same vertex set and the same bipartition is called a *revision of the ordered triple* (G,μ,e_0) if

$$d_H(v) = d_G(v) \ \text{ and } \ m_H(x,y) \le \mu(x,y)$$

for every vertex $v \in V(G) = X \cup Y$, every $x \in X$, and every $y \in Y$, and e_0 remains as an edge of H.

(III) Let H be a revision of the ordered triple (G,μ,e_0) and $e \in E(G)$. The edge e is *fixed* if the edge e remains as an edge of H.

Note that, according to Definition 2.2, parallel edges are only allowed to be incident with either x_0 or y_0 *whenever x_0y_0 is a special edge.*

Definition 2.3. Let G be a bipartite multigraph with the bipartition $\{X,Y\}$ and upper bounded by a function μ.

(I) A sequence $C = e_0e_1e_2e_3$ of $E(G)$ is called *an alternating 4-circuit of G* if
e_0, with the endvertices v_0 and v_1, is an edge of G,
e_1, with the endvertices v_1 and v_2, is NOT an edge of G,
e_2, with the endvertices v_2 and v_3, is an edge of G,
e_3, with the endvertices v_3 and v_0, is NOT an edge of G.

(For the convenience of discussion, sometimes, an alternating 4-circuit is denoted by its vertex sequence $v_0v_1v_2v_3v_0$ if

$$m_G(v_0,v_1) \ge 1, \quad m_G(v_2,v_3) \ge 1,$$
$$m_G(v_1,v_2) < \mu(v_1,v_2), \quad m_G(v_3,v_0) < \mu(v_3,v_0). \ )$$

(II) The *symmetric difference of the multigraph* G *and the alternating* 4-*circuit* C, denoted by $G\Delta C$, is the graph $[G\setminus\{e_0,e_2\}]\cup\{e_1,e_3\}$.

Definition 2.4. A bipartite multigraph G with the bipartition $\{X,Y\}$ is *complete* if $m_G(x,y)\geq 1$ for every $x\in X$ and every $y\in Y$.

2.3. Integer flows and circuit (cycle) covers

For a vertex v of a graph G, the set of edges of G incident with v is denoted by $E(v)$. If the edge set $E(G)$ is oriented, then the set of all arcs with tails (or heads) at a vertex v is denoted by $E^+(v)$ (or $E^-(v)$).

Definition 2.5. Let D be an orientation of a graph G and f be a function: $E(G)\mapsto Z$.
(I) The ordered pair (D,f) is called a *k-flow* of G if

$$0 \leq f(e) \leq k-1,$$

for every edge $e\in E(G)$, and

$$\sum_{e\in E^+(v)} f(e) \; = \; \sum_{e\in E^-(v)} f(e),$$

for ever vertex $v\in V(G)$.
(II) The *support* of a k-flow (D,f) of a graph G is the set of edges of G with $f(e)\neq 0$, and is denoted by *supp*(f).
(III) A k-flow (D,f) of G is *nowhere-zero* if $f(e)\neq 0$ for every edge e of G. (That is, *supp*$(f)=E(G)$.)

The concept of integer flow was originally introduced by Tutte [11], [12] (or see [14]) as a generalization of map coloring problems.

Definition 2.6. Let G be a graph.
(I) A family of cycles $\{C_1,\dots,C_t\}$ is called a *t-cycle cover* of G if every edge of G is contained in some member of $\{C_1,\dots,C_t\}$.
(II) A t-cycle cover $\{C_1,\dots,C_t\}$ of G is called a *t-cycle* $(1,2)$-*cover* of G if every edge of G is contained in precisely one or two members of $\{C_1,\dots,C_t\}$.
(III) A t-cycle cover $\{C_1,\dots,C_t\}$ of G is called a *t-cycle double cover* of G if every edge of G is contained in precisely two members of $\{C_1,\dots,C_t\}$.

Definition 2.7. Let $\{C_1,\dots,C_t\}$ be a t-cycle double cover of a graph G. If each cycle C_i can be oriented as a directed cycle such that every edge of G is contained in two members of $\{C_1,\dots,C_t\}$ with the opposite directions, then $\{C_1,\dots,C_t\}$ is called an *orientable t-cycle double cover* of G.

2.4. Partial Latin squares

In order to reduce the length of the paper, we will not present any definition about Latin squares since Theorem 1.2 is to be proved as a graph theory problem. Readers are referred to the article [5] or [6] for related definitions.

3. Lemmas for flows and cycle covers

Lemma 3.1. *If $\{C_1, C_2\}$ is a 2-cycle cover of a graph G and $e \in E(G)$, then (I) each of $\{C_1, C_2\}$, $\{C_1, C_1 \Delta C_2\}$, $\{C_1 \Delta C_2, C_2\}$ is a 2-cycle cover of G, and (II) one of $\{C_1, C_2\}$, $\{C_1, C_1 \Delta C_2\}$, $\{C_1 \Delta C_2, C_2\}$ covers the edge e only once.*

Proof. Obvious. ∎

Lemma 3.2. (See [14] Theorem 3.1.2) *Let G be a graph. The following statements are equivalent:*

(i) G admits a nowhere-zero 4-flow;
(ii) G has a 2-cycle cover;
(iii) G has a 3-cycle (1, 2)-cover.

Lemma 3.3. (Catlin, [3]; or, see [14] Lemma 3.8.11) *Let G be a graph and C be a circuit of G of length at most 4. Then G admits a nowhere-zero 4-flow if G admits a 4-flow (D, f) with $supp(f) \supseteq G \setminus E(C)$.*

Lemma 3.4. *If every edge of a graph G is contained in a circuit of length at most 4, then G admits a nowhere-zero 4-flow.*

Proof. By recursively contracting small circuits and applying Lemma 3.3. ∎

Lemma 3.5. (Hajiaghaee, Mahmoodian, Mirrokni, Saberi, and Tusserkani [7]) *The Keedwell-Cameron Conjecture is true if and only if every bipartite graphic sequence with the minimum degree at least 2 has a realization G such that G admits an orientable cycle double cover.*

Lemma 3.6. (Tutte [11], Jaeger [8], Archdeacon [1]; or, see [14] Theorem 3.6.1) *A graph G admits a nowhere-zero 4-flow if and only if G has an orientable 4-cycle double cover.*

4. Proof of the main theorem

Lemma 4.1. *Let C be an alternating 4-circuit of a bipartite multigraph G upper bounded by a function μ. If $e_0 \notin E(C)$, then $G \Delta C$ is a revision of (G, μ, e_0) (with e fixed, for every edge $e \in E(G) \setminus E(C)$).*

Proof. Obvious. ∎

The strategy and the outline of the proof. The proof of the main theorem is separated into two major steps. In the first step (Lemma 4.4), we are to show that the ordered triple (G,μ,e_0) has a revision such that the special edge e_0 is contained in a 4-circuit (except for an extreme structure). In the next step (Lemma 4.5), we are to recursively "contract" 4-circuits containing the special edge e_0 in a certain way, so that Lemma 3.3 can be applied in the inductive proof, and therefore some revision of (G,μ,e_0) admits a nowhere-zero 4-flow.

The following structure is an extreme case that will appear in the inductive proof of the main theorem.

Definition 4.2. (Structure $\mathcal{H}$) Let G be a bipartite multigraph upper bounded by a function μ with the bipartition $\{X,Y\}$ and with a special edge e_0 joining vertices x_0 and y_0. The ordered triple (G,μ,e_0) is the *structure* $\mathcal{H}$ if:

$$|X| \geq 2, \quad |Y| \geq 2,$$
$$\mu(x_0,y_0) = m_G(x_0,y_0) = 1,$$

and

$$m_G(x_0,y') \geq 2, \quad m_G(y_0,x') \geq 2, \quad m_G(x',y') = 0$$

for every $x' \in X \setminus \{x_0\}$ and every $y' \in Y \setminus \{y_0\}$.

The following lemma about the uniqueness of the revision of the structure $\mathcal{H}$, though is very easy to prove, will be applied later in proofs. From the following lemma, we can see that if an ordered triple (G,μ,e_0) is the structure $\mathcal{H}$, no revision of (G,μ,e_0) contains a 4-circuit passing through the special edge e_0.

Lemma 4.3. *Let G be a bipartite multigraph upper bounded by a function μ with $\delta(G)\geq 2$ and let $e_0 = x_0y_0$ be a special edge of G. If the ordered triple (G,μ,e_0) is the structure $\mathcal{H}$, then (G,μ,e_0) has only one revision; that is itself.*

Proof. Let H be a revision of the ordered triple (G,μ,e_0). Let $X' = X \setminus \{x_0\}$ and $Y' = Y \setminus \{y_0\}$. Since H is a revision of (G,μ,e_0) and $\mu(x_0,y_0) = m_G(x_0,y_0) = 1$, we have

$$\sum_{x\in X'} d(x) = d(y_0) - 1,$$

and

$$\sum_{y\in Y'} d(y) = d(x_0) - 1,$$

for BOTH G and H. Therefore, in both G and H, every vertex $x \in X'$ is adjacent to only y_0 with the same number of edges, and every vertex $y \in Y'$ is adjacent to only x_0 with the same number of edges. That is, $G = H$. ∎

Lemma 4.4. *Let G be a bipartite multigraph upper bounded by a function μ and with a special edge e_0 joining x_0 and y_0. If the minimum degree $\delta(G) \geq 2$ and the ordered triple (G, μ, e_0) is not the structure $\mathcal{H}$, then (G, μ, e_0) has a revision H such that*
(1) either e_0 is contained in a 4-circuit of H,
(2) or H admits a nowhere-zero 4-flow.

Proof. Prove by contradiction. Let G be a counterexample to the lemma such that

(i) $|E(G)|$ is as small as possible;

(ii) subject to (i), the component of G containing the special edge e_0 is as large as possible.

I. We claim that G *is connected.*

Let $Q_1, \ldots, Q_t$ be the components of G where $e_0 \in E(Q_1)$. Assume that $t \geq 2$. Since $\delta(G) \geq 2$, each component Q_i contains a circuit C_i of length at least 2. Let $e_i \in E(C_i) \setminus \{e_0\}$ for each $i = 1, 2$ with the endvertices x'_i $(\in X)$ and y'_i $(\in Y)$. Thus, we have an alternating 4-circuit $C = x'_1 y'_1 x'_2 y'_2 x'_1$. By Lemma 4.1, $H = G \Delta C$ is a revision of (G, μ, e_0). Furthermore, $(Q_1 \cup Q_2) \Delta C$ is a component (containing e_0) of the new graph H which is larger than Q_1 in G. This contradicts the choice of the counterexample (G, μ, e_0).

II. *Notations.* Let $N^k_G(v)$ (or simply, $N^k(v)$, if there is no confusion) be the set of all vertices u of $V(G) \setminus \{x_0, y_0\}$ such that the distance between v and u is k in $G \setminus [x_0, y_0]$ (where $[x_0, y_0]$ is the set of all edges joining x_0 and y_0).

For the sake of convenience, we denote,

$$X^1 = N^1_G(y_0) \ (\subseteq X), \quad Y^1 = N^1_G(x_0) \ (\subseteq Y),$$

$$X^2 = N^2_G(x_0) \ (\subseteq X), \quad Y^2 = N^2_G(y_0) \ (\subseteq Y).$$

III. *Case 1.* $X^2 \neq \emptyset$, and $Y^2 \neq \emptyset$. (See Figure 1.)

III-1. Since (G, μ, e_0) is a counterexample to the lemma, *no* 4*-circuit of G contains the edge e_0.* Thus,

$$[X^1, Y^1] = \emptyset, \quad [\{x_0\}, Y^2] = \emptyset, \text{ and } [\{y_0\}, X^2] = \emptyset.$$

III-2. We claim that *the subgraph of G induced by $X^2 \cup Y^2$ is a complete bipartite multigraph.*

Assume that $x_2 \in X^2$ and $y_2 \in Y^2$ with $m_G(x_2, y_2) = 0$. Let $x_1 \in N(y_2) \cap X^1$ and $y_1 \in N(x_2) \cap Y^1$. Then $C = y_1 x_2 y_2 x_1 y_1$ is an alternating 4-circuit of G

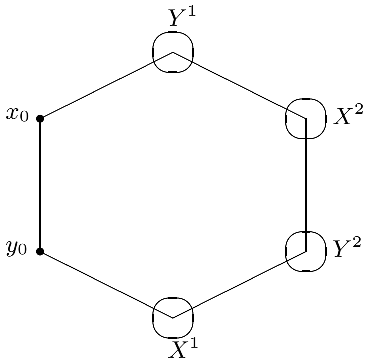

Figure 1. Case 1

since $m_G(x_1, y_1) = 0$ (by III-1). By Lemma 4.1, $H = G\Delta C$ is a revision of the ordered triple (G, μ, e_0) and furthermore, the revision H has a 4-circuit $y_0x_0y_1x_1y_0$ containing the edge e_0. This contradicts that the ordered triple (G, μ, e_0) is a counterexample.

III-3. We claim that $|Y^1| = 1$ (similarly, $|X^1| = 1$).

Assume that $|Y^1| > 1$. Let $y_2 \in Y^2$, $x_2 \in X^2$, $x_1 \in X^1 \cap N(y_2)$ and $y_1, y_1' \in Y^1$ with $y_1 \neq y_1'$ and $y_1' \in N(x_2)$. Here, $C = x_0y_1x_1y_2x_0$ is an alternating 4-circuit of (G, μ, e_0) since $m_G(y_1, x_1) = 0$ and $m_G(y_2, x_0) = 0$ (by III-1). By Lemma 4.1, the graph $H_1 = G\Delta C$ is a revision of (G, μ, e_0) and $x_0y_2 \in E(H_1)$. Furthermore, $C = y_0x_1y_1'x_2y_0$ is an alternating 4-circuit of (H_1, μ, e_0) since $m_G(y_1', x_1) = m_{H_1}(y_1', x_1) = 0$ and $m_G(x_2, y_0) = m_{H_1}(x_2, y_0) = 0$ (by III-1). By Lemma 4.1 again, the graph $H_2 = H_1\Delta C$ is a revision of (G, μ, e_0) and $x_0y_2, y_0x_2 \in E(H_2)$. Thus, H_2 contains a 4-circuit $y_0x_0y_2x_2y_0$ containing the edge e_0 since $y_2x_2 \in E(H_2)$ (by III-2). So, let

$$X^1 = \{x_1\} \text{ and } Y^1 = \{y_1\}.$$

III-4. It is obvious that *the subgraph G induced by $X^1 \cup Y^2$ (and $Y^1 \cup X^2$) is a complete bipartite graph* by III-3 and the definitions of X^2 and Y^2.

III-5. We claim that

$$N_G^3(x_0) \subseteq Y^2 \text{ and } N_G^3(y_0) \subseteq X^2.$$

Assume not. Let $y_3 \in N_G^3(x_0) \setminus Y^2$. That is, there is no edge of G joining y_3 and any vertex of X^1. Let $x_2 \in X^2 \cap N(y_3)$. Then, we have an alternating 4-circuit $C = y_0x_2y_3x_1y_0$ (where $\{x_1\} = X^1$ by III-3). By Lemma 4.1, $H = G\Delta C$ is a revision of (G, μ, e_0) and furthermore, the revision H has a 4-circuit $y_0x_0y_1x_2y_0$ containing the edge e_0 (where $\{y_1\} = Y^1$, by III-3). This contradicts that (G, μ, e_0) is a counterexample.

III-6. By III-5 and the definitions of X^k and Y^k, we can see that the subgraph of G induced by $\{x_0, y_0, x_1, y_1\} \cup X^2 \cup Y^2$ is a component of G. Since G is connected (by I), $\{x_0, y_0, x_1, y_1\} \cup X^2 \cup Y^2 = V(G)$.

III-7. Let G' be the graph obtained from G by deleting the vertices x_0 and y_0, and adding a new edge joining x_1 and y_1 (the only vertices of X^1 and Y^1, by III-3). It is obvious that the resulting graph G', by III-2 and III-4, is a complete bipartite graph with $\delta(G') \geq 2$. Therefore, G' admits a nowhere-zero 4-flow (by Lemma 3.4), so does G (applying Lemma 3.4 to some digons if $m_G(x_i, y_j) > 1$ for some $i, j \in \{0, 1\}$). This contradicts that (G, μ, e_0) is a counterexample and completes the proof of the lemma for this case. So, from now on, we will assume that

$$\text{either} \quad X^2 = \emptyset \quad \text{or } Y^2 = \emptyset.$$

IV. *Case 2.* $Y^1 = \emptyset$ (or $X^1 = \emptyset$). (See Figure 2.)

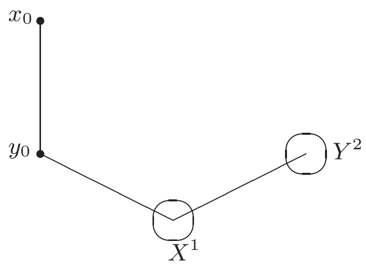

Figure 2. Case 2

IV-1. Since $Y^1 = \emptyset$, the vertex y_0 is the only neighbor of x_0. Hence,

$$m_G(x_0, y_0) \geq 2$$

because $\delta(G) \geq 2$.

IV-2. We claim that

$$N_G^3(y_0) = \emptyset.$$

Assume that $N_G^3(y_0) \neq \emptyset$. Let $x_3 \in N_G^3(y_0)$ and $y_2 \in Y^2 \cap N_G(x_3)$. Now, we have an alternating 4-circuit $C = x_0 y_0 x_3 y_2 x_0$. Then $H = G \Delta C$ is a revision of (G, μ, e_0) (note, the edge joining x_0 and y_0 in C is not the edge e_0 since, by IV-1, $m_G(x_0, y_0) \geq 2$). Furthermore, there is a 4-circuit $x_0 y_0 x_1 y_2 x_0$ in H containing the special edge e_0. This contradicts that the ordered triple (G, μ, e_0) is a counterexample.

IV-3. By IV-2 and the definitions of X^k and Y^k,

$$V(G) = \{x_0, y_0\} \cup X^1 \cup Y^2.$$

IV-4. Now, we claim that *every edge of G is contained in a circuit of length* ≤ 4. We only need to consider $xy \in E(G)$ with $m_G(x,y)=1$.

(i) By IV-1, every edge between x_0 and y_0 is in a 2-circuit.

(ii) Note that $\mu(x_1,y_2)=1$ for every $x_1 \in X^1$ and every $y_2 \in Y^2$ (by Definition 2.2.II). Since $\delta(G)\geq 2$, the vertex $y_2 \in Y^2$ has at least two distinct neighbors x_1, x_1' in X^1. Thus, the edge x_1y_2 of $[X^1, Y^2]$ is contained in a 4-circuit $y_0x_1y_2x_1'y_0$.

(iii) For each $x_1 \in X^1$ with $m_G(y_0,x_1)=1$, we have $N(x_1)\cap Y^2 \neq \emptyset$ since $\delta \geq 2$. Similar to (ii), the edge y_0x_1 is contained a 4-circuit $y_0x_1y_2x_1'y_0$ of G where $y_2 \in Y^2$ and $x_1, x_1' \in N(y_2)$.

So, by Lemma 3.4, G itself admits a nowhere-zero 4-flow. This contradicts that the ordered triple (G,μ,e_0) is a counterexample to the lemma and completes the proof for this case. So, from now on, we will assume that

$$X^1 \neq \emptyset \neq Y^1.$$

V. *Case 3.* $X^2=\emptyset$ (or $Y^2=\emptyset$). (See Figure 3.)

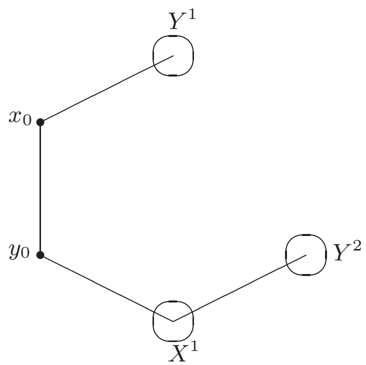

Figure 3. Case 3

V-1. Note that, by IV (Case 2),

$$X^1 \neq \emptyset \quad \text{and} \quad Y^1 \neq \emptyset.$$

Similar to III-1,

$$[X^1, Y^1] = \emptyset \text{ and } [\{x_0\}, Y^2] = \emptyset.$$

V-2. Since $X^2=\emptyset$ and $Y^1 \neq \emptyset$, x_0 is the only neighbor of every vertex $y_1 \in Y^1$. Since $\delta(G)\geq 2$, we have

$$m_G(x_0, y_1) \geq 2$$

for every $y_1 \in Y^1$.

V-3. *Subcase 3-1.* $Y^2 \neq \emptyset$.

Since $Y^2 \neq \emptyset$ in this subcase, let $y_2 \in Y^2$ and $x_1 \in X^1 \cap N(y_2)$, and let $y_1 \in Y^1$. Then $C = y_2x_0y_1x_1y_2$ is an alternating 4-circuit of G. By Lemma 4.1, $H = G\Delta C$ is a revision of (G,μ,e_0) and furthermore, there is a new edge joining x_1 and y_1 in H while there remains an edge joining x_0 and y_1 since $m_G(x_0,y_1) \geq 2$ (by V-2). Thus, $y_0x_0y_1x_1y_0$ is a 4-circuit of H containing the special edge e_0. This contradicts that the ordered triple (G,μ,e_0) is a counterexample.

V-4. *Subcase 3-2.* $Y^2 = \emptyset$. (We are to show that (G,μ,e_0) is the structure $\mathcal{H}$ in this subcase.)

Now, both X^2 and Y^2 are empty. By V-2 and similar to V-2,

$$m_G(x_0,y_1) \geq 2 \quad \text{and} \quad m_G(y_0,x_1) \geq 2$$

for every $x_1 \in X^1$ and every $y_1 \in Y^1$.

If $m_G(x_0,y_0) \geq 2$, then every edge of G is contained in a 2-circuit of G. By Lemma 3.4, G admits a nowhere-zero 4-flow. So, we have

$$m_G(x_0,y_0) = 1.$$

If $\mu(x_0,y_0) \geq 2$, then $C = x_0y_0x_1y_1x_0$ is an alternating 4-circuit of G (where the edge of C joining x_0 and y_0 is not an edge of G since $\mu(x_0,y_0) > m_G(x_0,y_0) = 1$). By Lemma 4.1, $H = G\Delta C$ is a revision of (G,μ,e_0). Note that, $m_H(x_0,y_1) = m_G(x_0,y_1) - 1 \geq 1$ and $m_H(y_0,x_1) = m_G(y_0,x_1) - 1 \geq 1$. Thus, the special edge e_0 is contained in a 4-circuit $x_0y_0x_1y_1x_0$ of H. This contradicts that the ordered triple (G,μ,e_0) is a counterexample. Hence, we have

$$\mu(x_0,y_0) = 1.$$

Now, it is obvious that the ordered triple (G,μ,e_0) is the structure $\mathcal{H}$ and the proof of the lemma is therefore completed. ∎

Lemma 4.5. *Let G be a bipartite multigraph upper bounded by a function μ and with a special edge e_0 joining x_0 and y_0. If the minimum degree $\delta(G) \geq 2$ and (G,μ,e_0) is not the structure $\mathcal{H}$, then (G,μ,e_0) has a revision H that admits a nowhere-zero 4-flow.*

Proof. Let (G,μ,e_0) be a counterexample to the lemma with the least number of edges.

I. Since the ordered triple (G,μ,e_0) is not the structure $\mathcal{H}$, by Lemma 4.3, no revision of (G,μ,e_0) is the structure $\mathcal{H}$.

By Lemma 4.4, we may assume that the special edge e_0 is contained in some 4-circuit $C=x_0y_0x_1y_1x_0$ of G (with the edges e_0 joining x_0 and y_0, e_1 joining y_0 and x_1, e_2 joining x_1 and y_1 and e_3 joining y_1 and x_0,)

II. We claim that G *is connected.* Assume that $Q_1,\ldots,Q_t$ are the components of G with $t\geq 2$ and $e_0\in Q_1$. Since each Q_i is smaller than G, the lemma holds for each component.

Since the special edge e_0 is not contained in any Q_i for $i\geq 2$ and, by the definitions of the special edge and the upper bound function μ (Definition 2.2), we have $\mu(x,y)=1$ for every $x\in X\cap Q_i\subseteq X\setminus\{x_0\}$ and $y\in Y\cap Q_i\subseteq Y\setminus\{y_0\}$. Therefore, none of $Q_2,\ldots,Q_t$ is the structure $\mathcal{H}$ (by Definition 4.2) and, hence, each of them has a revision H_i admitting a nowhere-zero 4-flow.

For the component Q_1 that contains the special edge e_0, it is obvious that Q_1 is not the structure $\mathcal{H}$ since e_0 is contained in a 4-circuit of Q_1 (while the structure $\mathcal{H}$ does not contain any 4-circuit, by Definition 4.2). So, (Q_1,μ,e_0) has a revision H_1 that admits a nowhere-zero 4-flow. Put all $H_1,\ldots,H_t$ together, (G,μ,e_0) has a revision $H_1\cup\ldots\cup H_t$ that admits a nowhere-zero 4-flow.

III. Let G^* be the graph obtained from G by identifying $\{x_0,x_1\}$ to be a new vertex x^*, identifying $\{y_0,y_1\}$ to be a new vertex y^* and deleting the edges e_1,e_2 and e_3 (note that $x_0y_0x_1y_1x_0$ is the 4-circuit defined in subsection I). Also define μ^* for G^* as follows:

$\mu^*(x,y)=\mu(x,y)$ if $x,y\notin\{x^*,y^*\}$, and
$\mu^*(x^*,y)=\mu(x_0,y)+\mu(x_1,y)$ if $y\neq y^*$, and
$\mu^*(x,y^*)=\mu(x,y_0)+\mu(x,y_1)$ if $x\neq x^*$, and
$\mu^*(x^*,y^*)=\mu(x_0,y_0)+\mu(y_0,x_1)+\mu(x_1,y_1)+\mu(y_1,x_0)-3.$

Note that, in the new graph G^*, the edge e_0 joining x^* and y^* remains to be the special edge. (See Figure 4.)

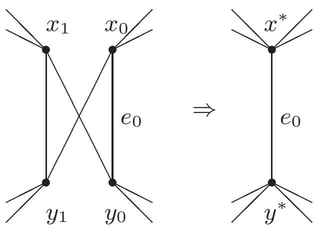

Figure 4.

Since (G,μ,e_0) is a smallest counterexample to the lemma, either some revision of (G^*,μ^*,e_0) admits a nowhere-zero 4-flow, or $\delta(G^*) < 2$, or (G^*,μ^*,e_0) is the structure $\mathcal{H}$.

IV. Assume that $\delta(G^*) \geq 2$ and (G^*,μ^*,e_0) is not the structure $\mathcal{H}$. Thus, (G^*,μ^*,e_0) has a revision H^* and H^* admits a nowhere-zero 4-flow (D^*,f^*). One can construct a revision H of (G,μ,e_0) from H^* as follows: splitting the vertices x^* and y^* back to the vertices $\{x_0,x_1\}$ and $\{y_0,y_1\}$ respectively, and adding the edges e_1, e_2 and e_3 back such that $d_H(v)=d_G(v)$ for every $v \in X \cup Y$.

By Lemma 3.3, the 4-flow (D^*,f^*) of H^* can be extended to the entire graph H since the support of any trivial extension of (D^*,f^*) from H^* to the new graph H covers every edge of $H \setminus E(C)$.

V. We claim that (G^*,μ^*,e_0) *is not the structure* $\mathcal{H}$.

Otherwise, by Lemma 3.4, $G^* \setminus \{e_0\}$ admits a nowhere-zero 4-flow since every edge of $G^* \setminus \{e_0\}$ is a parallel edge and furthermore, by Lemma 3.3, this 4-flow can be extended to the entire graph G.

VI. By IV and V, we have

$$\delta(G^*) < 2.$$

Note that, during the construction of G^* from G (see III), the operations of vertex identifications and edge deletions occur only at the vertices x^* and y^*. That is, either $d_{G^*}(x^*)=1$ or $d_{G^*}(y^*)=1$.

VII. It is impossible that $d_{G^*}(x^*) = d_{G^*}(y^*) = 1$. For otherwise, the 4-circuit C is a component of G. By II, G is connected. So, $G = C$ is a 4-circuit that admits a nowhere-zero 2-flow. This contradicts that (G,μ,e_0) is a counterexample.

VIII. Without loss of generality, by VI and VII, we have

$$d_{G^*}(x^*) = 1 \text{ and } d_{G^*}(y^*) > 1.$$

Assume that $d_{G^*}(y^*) \geq 3$. Let $G^{**} = G^* \setminus \{x^*\}$. Here, $\delta(G^{**}) \geq 2$. Note that, in G^*, $\mu^*(x,y) > 1$ only if $y = y^*$. One can choose any edge of G^{**} incident with y^*, say e_1 (joining y^* and x_2), as the new special edge of G^{**}.

If the ordered triple (G^{**},μ^*,e_1) is the structure $\mathcal{H}$, then, by the definition of the structure $\mathcal{H}$, we have

$$|X(G^*) \cap G^{**}| = |X(G^*) \setminus \{x^*\}| \geq 2,$$

$$|Y(G^*) \cap G^{**}| = |Y(G^*)| \geq 2$$

and

$$m_{G^{**}}(x_2, y) \geq 2$$

for every $y \in Y(G^*) \setminus \{y^*\} = Y \setminus \{y_0, y_1\}$. But, in the original graph G, $m_G(x_2, y) = 1$ since $x_2 \neq x_0$ and $y \in Y \setminus \{y_0\}$. This contradicts that $m_G(x_2, y) = m_{G^{**}}(x_2, y)$ and therefore, (G^{**}, μ^*, e_1) is not the structure $\mathcal{H}$.

Since the ordered triple

(G, μ, e_0) is a smallest counterexample to the lemma and G^{**} is smaller than G, (G^{**}, μ^*, e_1) has a revision H^{**} that admits a nowhere-zero 4-flow. This 4-flow can be extended to G by Lemma 3.3. So

$$d_{G^*}(y^*) = 2.$$

IX. Finally, the following is the only remaining case

$$d_{G^*}(x^*) = 1 \quad \text{and} \quad d_{G^*}(y^*) = 2.$$

Let $P = x^* y^* \ldots w$ be a longest subdivided edge of G^* containing the edge $e_0 = x^* y^*$. (See Figure 5.)

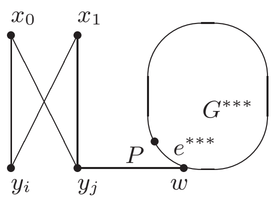

Figure 5.

Let $G^{***} = G^* \setminus [V(P) \setminus \{w\}]$. Here, $\delta(G^{***}) \geq 2$. Since G^{***} does not contain x^* or y^*, we have $\mu^* = 1$ everywhere in G^{***} and therefore, for any edge e of G^{***}, the ordered triple (G^{***}, μ^*, e) cannot be the structure $\mathcal{H}$. Furthermore, one may choose an edge $e^{***} = x_3 y_3$ of G^{***} incident with w (that is, $w \in \{x_3, y_3\}$) as the new special edge of the ordered triple $(G^{***}, \mu^*, e^{***})$. So, the ordered triple $(G^{***}, \mu^*, e^{***})$ has a revision H^{***} and admitting a nowhere-zero 4-flow. By Lemma 3.2, H^{***} has a 2-cycle cover $\{C_1, C_2\}$. Furthermore, by Lemma 3.1, each of $\{C_1, C_2\}$, $\{C_1, C_1 \Delta C_2\}$ and $\{C_1 \Delta C_2, C_2\}$ is a 2-cycle cover of H^{***}, and one of $\{C_1, C_2\}$, $\{C_1, C_1 \Delta C_2\}$ and $\{C_1 \Delta C_2, C_2\}$ covers the edge e^{***} precisely once, say, $\{C_1, C_2\}$ is such a 2-cycle cover of H^{***} that $e^{***} \in C_1 \setminus C_2$.

Since $d_{G^*}(y^*) = 2$, we have

$$d_G(y_i) = 2 \quad \text{and} \quad d_G(y_j) = 3$$

for some $\{i,j\} = \{0,1\}$. Here, we have an alternating 4-circuit $C^* = y_i x_1 y_3 x_3 y_i$. Thus, $H = [H^{***} \cup P \cup C] \Delta C^*$ is a revision of (G, μ, e_0) and $\{C_3 = C_1 \Delta C^* \Delta C, C_2\}$ is a family of cycles of H that every edge of $[C^* \Delta C] \setminus \{e^{***}\}$ is covered by C_3 but not C_2. Hence, $\{C_3, C_2\}$ covers every edge of H but not any of $E(P \setminus \{x^*\})$ (note that $P \setminus \{x^*\}$ is a path joining w and y_j in G). Let $C_4 = P \cup Q$ where Q is a segment of $[C^* \Delta C] \setminus \{e^{***}\}$ joining $w \in \{x_3, y_3\}$ and y_j. Then $\{C_2, C_3, C_4\}$ is a 3-cycle (1, 2)-cover of G. By Lemma 3.2, the revision H admits a nowhere-zero 4-flow. This contradicts that the ordered triple (G, μ, e_0) is a counterexample and therefore completes the proof of the lemma. ∎

Proof of Theorem 1.1. By the definition of realizations of a graphic sequence, parallel edges are not allowed in any realization. Let G be a realization of a given bipartite graphic sequence with the minimum degree $\delta(G) \geq 2$ and let $\mu(x,y) = 1$ (initially) for every $x \in X$ and every $y \in Y$. Thus, the ordered triple (G, μ, e_0) cannot be the structure $\mathcal{H}$. By Lemma 4.5, the ordered triple (G, μ, e_0) has a revision H which remains as a realization of S and admits a nowhere-zero 4-flow. ∎

Proof of Theorem 1.2. This is a corollary of Theorem 1.1, Lemma 3.6 and Lemma 3.5. ∎

5. Remarks

Readers may wonder whether Theorem 1.1 holds for general graphs. The following theorem is the counterpart for general graphs.

Theorem 5.1. *Let S be a graphic sequence with the minimum degree $\delta(S) \geq 2$. Then S has a realization G that admits a nowhere-zero 4-flow.*

The proof of Theorem 1.1 can be adapted for this theorem. However, without the restriction of bipartiteness in our reduction processing, the present proof becomes straightforward and much easier.

Proof. Let G be a counterexample to the theorem with the least number of vertices.

I. If G contains a circuit of length at most 4, then, contracting this small circuit and recursively contracting all resulting small circuits (of length ≤ 4), the resulting graph G' remains simple and is smaller than G. Hence, G' has a revision G'' admitting a nowhere-zero 4-flow. By Lemma 3.3, the 4-flow of G'' can be extended to a revision of G. So, the girths of G and all of its revisions are at least 5.

II. Let $P = v_0 \dots v_p$ be a longest path of G. Assume that $p \geq 5$. By I, $v_i v_{i+\mu} \notin E(G)$ for each $i = 0, \dots, p-4$ and $2 \leq \mu \leq 3$. We are to show that $v_0 v_4 \in E(G)$. If not, then $C = v_0 v_1 v_3 v_4 v_0$ is an alternating 4-circuit of G. By Lemma 4.1, $G \Delta C$ is a revision of G and contains a triangle $v_1 v_2 v_3 v_1$. This contradicts I. With the same argument, one can prove that $v_1 v_5 \in E(G)$. Now, $v_0 v_1 v_5 v_4 v_0$ is a 4-circuit of G, which contradicts I again.

III. It remains to consider the case $p \leq 4$. Since P is a longest path, all neighbors of v_0 are contained in P. Furthermore, v_0 has a neighbor v_i with $i > 1$ since $\delta \geq 2$. It is not hard to see that $p \geq i \geq 4$ because of I. So, $v_i = v_p$ and $p = 4$ as $p \leq 4$. Note that P induces a longest circuit (5-circuit) $C = v_0 \dots v_4 v_0$. By applying the above argument to each longest path $v_i C v_{i-1}$, we have $d(v_i) = 2$ for every $i \in Z_5$, and therefore, $G = C$. The 5-circuit G admits a nowhere-zero 2-flow. This contradicts that G is a counterexample. ∎

We close by suggesting one research problem, which is closely related to the main theorems.

Problem 5.2. Characterize all graphic sequences S that no realization of S admits a nowhere-zero 3-flow.

One may pay attention only to graphic sequences with $\delta \geq 3$, since degree 2 vertices can be created by inserting new vertices into some edges.

Note that some graphic sequences do not have any realization which admits nowhere-zero 3-flow. For example, the degree sequences of all odd-wheels: $S = \{k, 3^k\}$ (where k is an odd number).

References

[1] D. Archdeacon: Face coloring of embedded graphs, *J. Graph Theory* 8 (1984), 387–398.

[2] P. J. Cameron: (BCC 16.15/DM 328) Problems from the 16th British Combinatorial Conference, *Discrete Math.* **197/198** (1999), 799–812.

[3] P. A. Catlin: Double cycle covers and the Petersen graph, *J. Graph Theory* **13** (1989), 465–483.

[4] D. A. Gale: A theorem on flows in networks, *Pac. J. Math.* **7** (1957), 1073–1082.

[5] A. D. Keedwell: Critical sets and critical partial Latin squares, *Combinatorics, graph theory, algorithms and applications* (Beijing, 1993), World Sci. Publishing, River Edge, NJ. (1994), 111–123.

[6] A. D. Keedwell: Critical sets for Latin squares, graphs and block designs: a survey; *Festschrift for C. St. J. A. Nash-Williams*, *Congr. Numer.* **113** (1996), 231–245.

[7] M. T. Hajiaghaee, E. S. Mahmoodian, V. S. Mirrokni, A. Saberi and R. Tusserkani: On the simultaneous edge-coloring conjecture, *Discrete Math.* **216(1–3)** (2000), 267–272.

[8] F. Jaeger: Flows and generalized coloring theorems in graphs, *J. Combin. Theory Ser. B* **26** (1979), 205–216.
[9] M. Mahdian, E. S. Mahmoodian, A. Saberi, M. R. Salavatipour and R. Tusserkani: On a conjecture of Keedwell and the cycle double cover conjecture, *Discrete Math.* **216(1–3)** (2000), 287–292.
[10] H. J. Ryser: Matrices of zeros and ones, *Canad. J. Math.* **9** (1957), 371–377.
[11] W. T. Tutte: On the imbedding of linear graphs in surfaces, *Proc. London Math. Soc., Ser. 2* **51** (1949), 474–483.
[12] W. T. Tutte: A contribution on the theory of chromatic polynomial, *Canad. J. Math.* **6** (1954), 80–91.
[13] D. B. West: *Introduction of Graph Theory*, Prentice Hall, NJ. 1996.
[14] C. Q. Zhang: *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, New York, 1997.

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